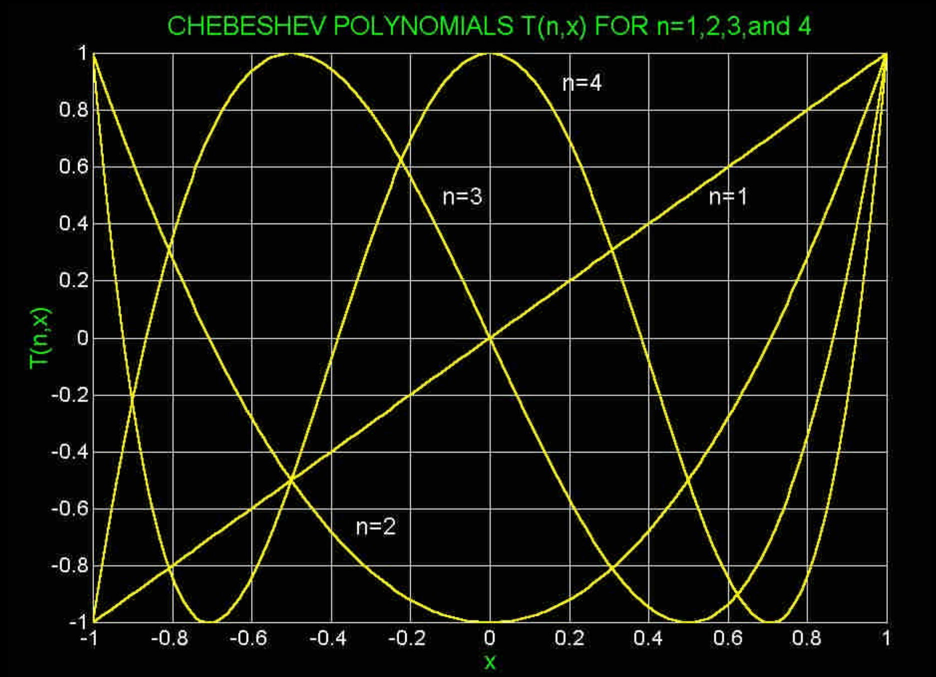
Function Approximation



Pafnuty Chebyshev, 1821-1894

Administration

- Assignment 3 is out; due on April 23 midnight
- Chapter 3 Excises 30 (a)(b)(c)(d), 33, 48, 49, 69 (20% each)
- Extra credit (10%): App 8
- Mid-term exam: Apr 30

Function Approximation

- How does a computer approximate *cos*(*x*), *exp*(*x*), and other non-polynomial functions?
- Efficient and accurate approximation is desired
  - Chebyshev polynomials
    - Orthogonal polynomials
    - Converge faster than a Taylor series
  - Rational functions
    - Ratio of two polynomials
  - Fourier series
    - Series of sine and cosine terms
    - Can approximate periodic or discontinuous functions

Truncation Error in Taylor Series

- Recall that Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= f(a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

- If approximating a function using first *n* terms

$$Error = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

- Error grows rapidly as x-value departs from *x = a!*

Chebyshev Polynomials

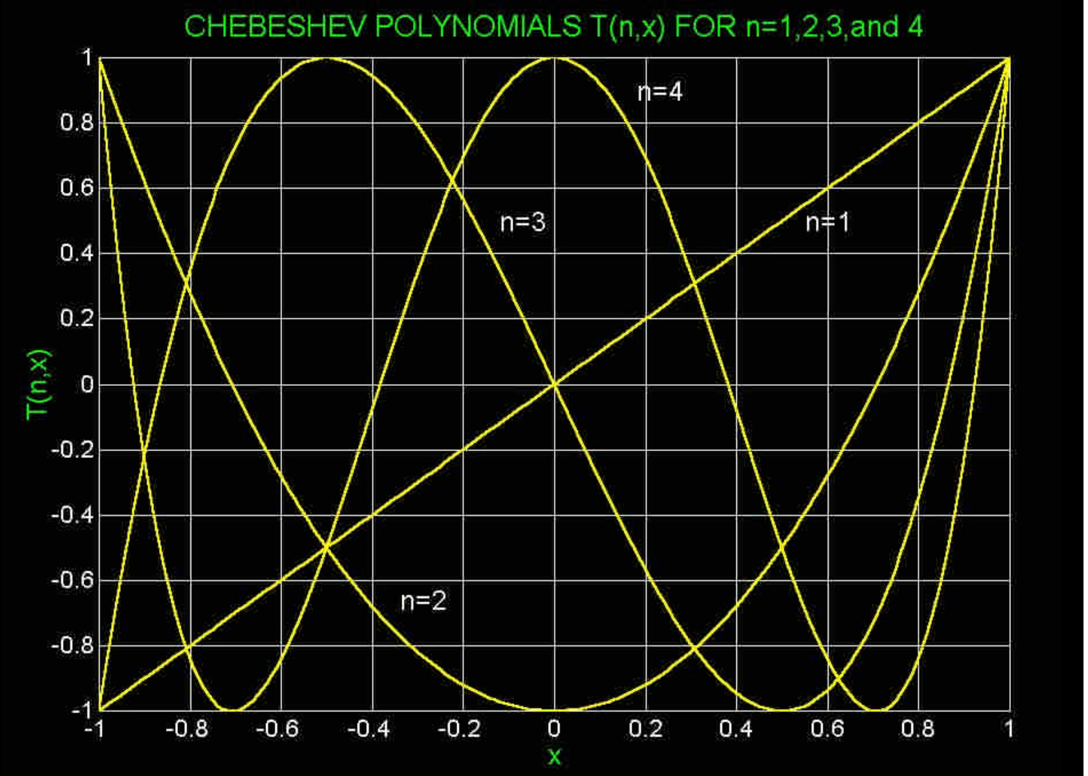
- Recursive definition

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 1, \qquad T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$



Chebyshev Polynomials (cont.)

- Equivalent form

$$T_n(x) = \cos(n \cdot \mathrm{acos}(x)) \qquad x = \cos(\theta)$$

- Proof

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta \cos\theta$$

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 1, \qquad T_1(x) = x$$

Chebyshev Series

$$T_0(x) = 1 \qquad 1 = T_0$$

$$T_1(x) = x \qquad x = T_1$$

$$T_2(x) = 2x^2 - 1 \qquad x^2 = \frac{1}{2}(T_2 + T_0)$$

$$T_3(x) = 4x^3 - 3x \qquad x^3 = \frac{1}{4}(T_3 + 3T_1)$$

$$T_4(x) = 8x^4 - 8x^2 + 1 \qquad x^4 = \frac{1}{8}(T_4 + 4T_2 + 3T_0)$$

Chebyshev Series is More Economical

- On the interval [-1, 1], approximating a function with a series of Chebyshev polynomials is better than with a Tayor series as it has a smaller maximum error

- On an interval [*a*, *b*], we have to transform the function to translate the *x*-value into [-1, 1]

$$f(x) \rightarrow f(\tilde{x}) \qquad \tilde{x} = 2\frac{x-a}{b-a} - 1$$

Example: Economizing a Power Series

- Maclaurin series is Taylor series expansion of a function about 0
- Maclaurin series of e^x

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots$$

$$x^2 = \frac{1}{2}(T_2 + T_0)$$

$$x^3 = \frac{1}{4}(T_3 + 3T_1)$$

$$x^4 = \frac{1}{8}(T_4 + 4T_2 + 3T_0)$$

- Approximated by a Chebyshev series

$$e^x \approx T_0 + T_1 + \frac{1}{2}\cdot\frac{1}{2}(T_2 + T_0) + \frac{1}{6}\cdot\frac{1}{4}(T_3 + 3T_1) + \frac{1}{24}\cdot\frac{1}{8}(T_4 + 4T_2 + 3T_0)$$

$$+ \frac{1}{120}\frac{1}{16}(10T_1 + 5T_3 + \cdots) + \frac{1}{23040}(10T_0 + 15T_2 + \cdots) + \cdots$$

Example (cont.)

- Maclaurin series of e^x truncated after 3rd degree

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots$$

$$\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

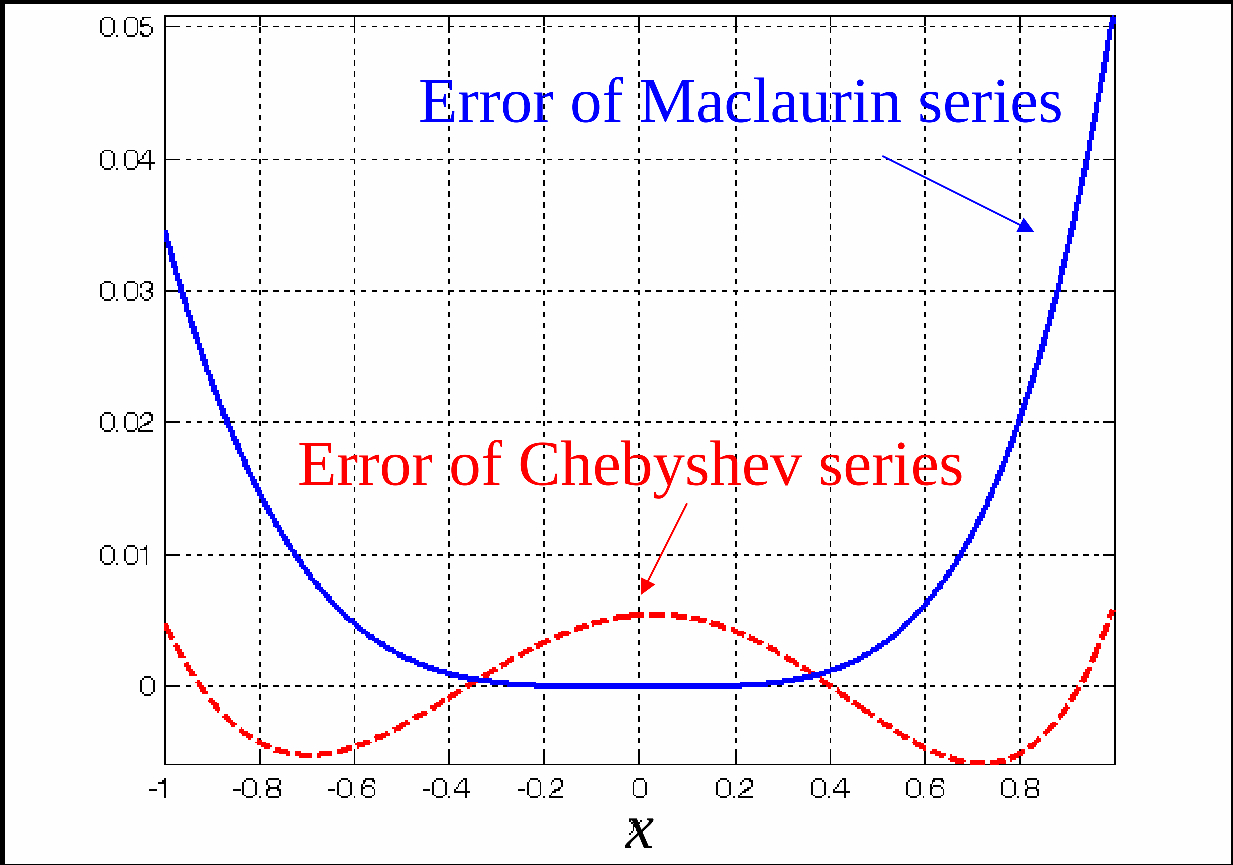
- Chebyshev series of e^x truncated after 3rd degree

$$e^x = 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \cdots$$

$$\approx 0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3$$

Example (cont.)

- Error of Maclaurin series grows rapidly when *x-value* departs from zero



Orthogonaliy of Chebyshev Polynomials

- Two polynomials $p(x)$, $q(x)$ are orthogonal on an interval $[a, b]$ if their inner product

$$< p,q >= \int_a^b p(x)q(x)w(x)dx = 0, \qquad w(x) \geq 0$$

- Set of polynomials is orthogonal if each polynomial is orthogonal to each other

- Chebyshev Polynomials are orthogonal on interval [-1, 1] with $w(x) = 1 \big/ \sqrt{1 - x^2}$

Verify Orthogonaliy of Chebyshev Polynomials

- Example: $p(x) = T_0(x)$, $q(x) = T_3(x)$

$$< p(x), q(x) > \int_{-1}^{1} 1 \cdot (4x^3 - 3x) \frac{1}{\sqrt{1-x^2}} dx = 0$$

- Matlab code
  - int('(4*x^3-x)/sqrt(1-x^2)', -1 ,1)

Least Squares Approximation using Orthogonal Polynomials

- System of normal equations for a high-degree polynomial fit is ill-conditioned
- Fitting data with orthogonal polynomials such as Chebyshev polynomials can reduce the condition number to about 5

$$f(x) = a_0 T_0(x) + a_1 T_1(x) + \cdots + a_n T_n(x)$$

$$\begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_n(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_m) & T_1(x_m) & \cdots & T_n(x_m) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Rational Function Approximation

- Approximating a function with a Chebyshev series has smaller maximum error than with a Taylor series

- We can further improve the maximum error using rational function approximation

$$f(x) \approx R_N(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_m x^m}$$

$$N = n + m + 1$$

- Padé approximation: $n \geq m$

Padé Approximation

- Represent a function in a Taylor series expanded at $x = 0$
- Coefficients are determined by setting

$$f(x) = R_N(x) \qquad c_i = \frac{f^{(i)}(0)}{i!}$$

$$(c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_m x^m}$$

and equating coefficients

$$a_0 = c_0$$

$$a_1 = b_1 c_0 + c_1$$

$$\vdots$$

$$a_n = b_m c_{n-m} + \cdots + b_1 c_{n-1} + c_n$$

$$c_{n+1} + b_1 c_n + \cdots + b_m c_{n-m+1} = 0$$

$$\vdots$$

$$c_N + b_1 c_{N-1} + \cdots + b_m c_{N-m} = 0$$

Example: Padé Approximation

- Find $\arctan(x) \approx R_{10}(x)$
- Maclaurin series of $\arctan(x)$ through x^{10} is

$$\arctan(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$$

- $f(x) = R_{10}(x)$

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5}$$

Example: Padé Approximation (cont.)

- Equating coefficients

$$x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9=\frac{a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5}{1+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5}$$

- x^0 through x^5

$$a_0=0$$
$$a_1=1$$
$$a_2=b_1$$
$$a_3=-\tfrac{1}{3}+b_2$$
$$a_4=-\tfrac{1}{3}b_1+b_3$$
$$a_5=\tfrac{1}{5}-\tfrac{1}{3}b_2+b_4$$

- x^6 through x^{10}

$$\tfrac{1}{5}b_1-\tfrac{1}{3}b_3=0$$
$$-\tfrac{1}{7}+\tfrac{1}{5}b_2-\tfrac{1}{3}b_4=0$$
$$-\tfrac{1}{7}b_1+\tfrac{1}{5}b_3-\tfrac{1}{3}b_5=0$$
$$\tfrac{1}{9}-\tfrac{1}{7}b_2+\tfrac{1}{5}b_4=0$$
$$\tfrac{1}{9}b_1-\tfrac{1}{7}b_3+\tfrac{1}{5}b_5=0$$

Padé Approximation vs. Maclaurin Series

- Padé Approximation

$$\arctan(x) \approx \frac{x + \frac{7}{9}x^3 + \frac{64}{945}x^5}{1 + \frac{10}{9}x^2 + \frac{5}{21}x^4}$$

- Maclaurin series

$$\arctan(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$$

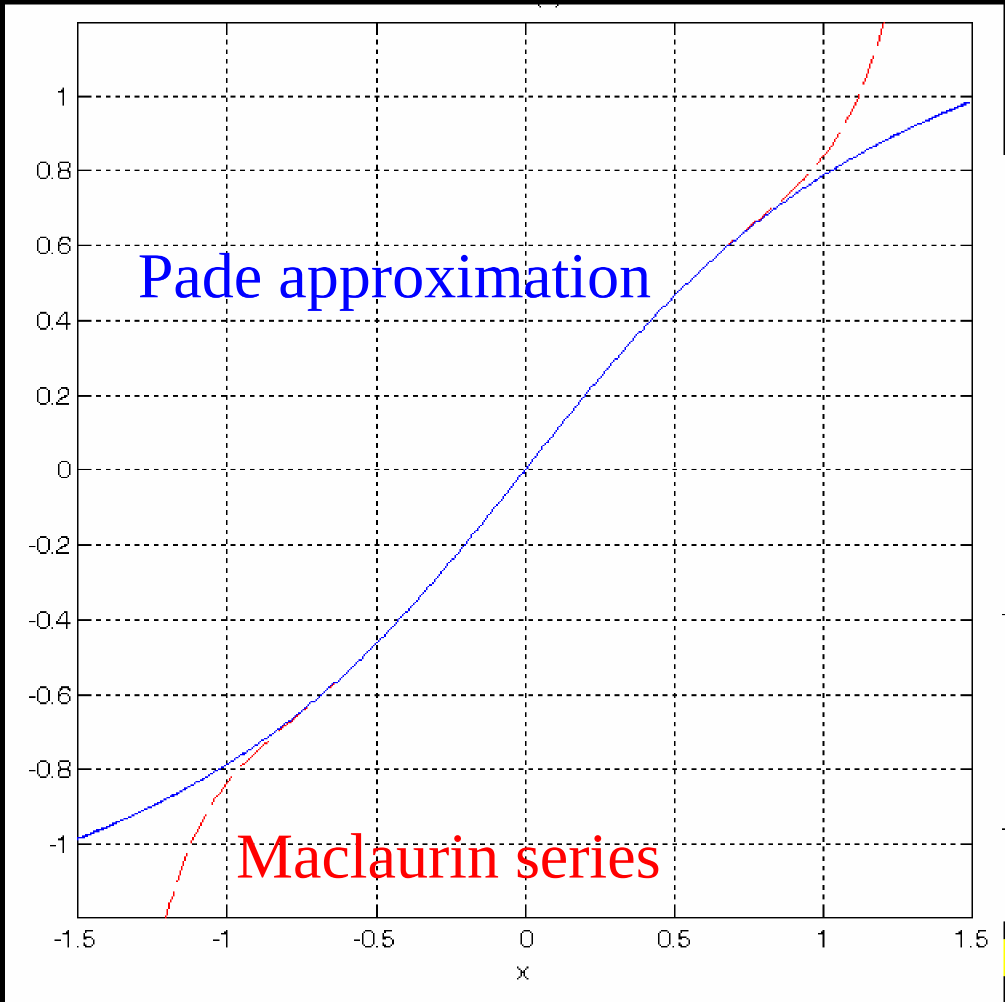
Table 4.4 Comparison of Padé approximation to Maclaurin series for arctan x

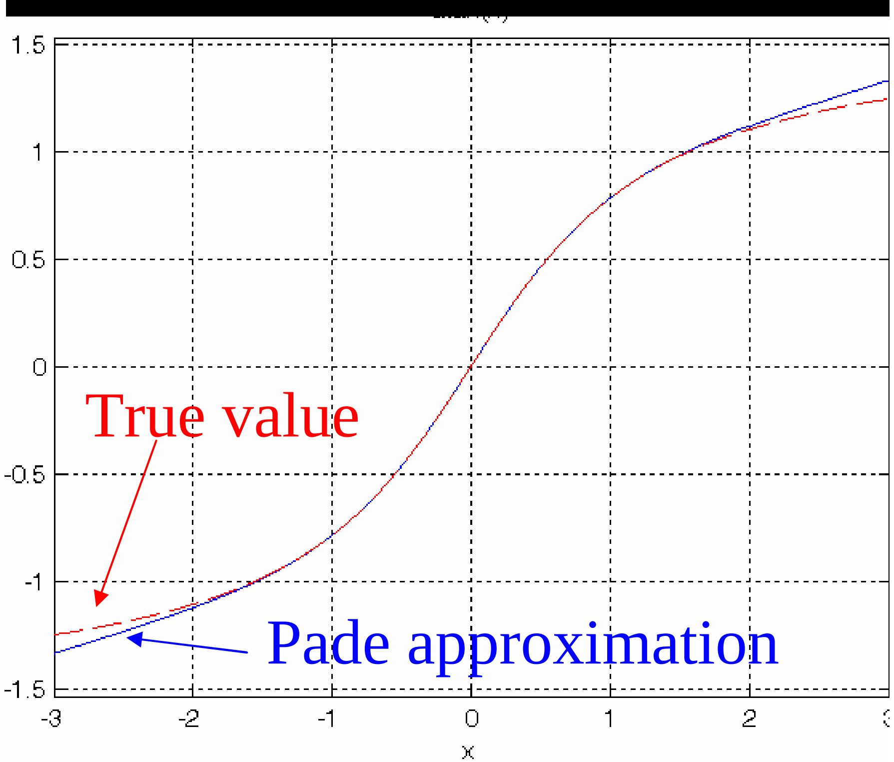
x	True value	Padé (Eq. 4.12)	Error	Maclaurin (Eq. 4.10)	Error
0.2	0.19740	0.19740	0.00000	0.19740	0.00000
0.4	0.38051	0.38051	0.00000	0.38051	0.00000
0.6	0.54042	0.54042	0.00000	0.54067	−0.00025
0.8	0.67474	0.67477	−0.00003	0.67982	−0.00508
1.0	0.78540	0.78558	−0.00018	0.83492	−0.04952

Padé Approximation vs. Maclaurin Series

$$\arctan(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$$

$$\arctan(x) \approx \frac{x + \frac{7}{9}x^3 + \frac{64}{945}x^5}{1 + \frac{10}{9}x^2 + \frac{5}{21}x^4}$$





Chebyshev-Padé Rational Function

- We can get a rational function approximation from a Chebyshev series expansion of a function as well
- Example:

$$e^x = 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \cdots$$

- We first form

$$1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 - \frac{a_0 + a_1T_1 + a_2T_2}{1 + b_1T_1}$$

- Then expand the numerator and set the coefficients of each degree of the *T*'s to zero

- Recall that $T_n(x) = \cos(n\theta)$

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta]$$

$$T_n(x)T_m(x) = \frac{1}{2}[T_{n+m}(x) + T_{|n-m|}(x)]$$

- Apply the last equation to

$$1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 - \frac{a_0 + a_1T_1 + a_2T_2}{1 + b_1T_1}$$

- The numerator becomes

$$1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \frac{1.2661b_1}{2}(T_1 + T_1)$$

$$+\frac{1.1302b_1}{2}(T_2 + T_0) + \frac{0.2715b_1}{2}(T_3 + T_1) + \frac{0.0443b_1}{2}(T_4 + T_2)$$

$$= a_0 + a_1T_1 + a_2T_2$$

Equating the Coefficients of Each Degree of *T*'s in Numerator

$$a_0 + a_1T_1 + a_2T_2 =$$

$$1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \frac{1.2661b_1}{2}(T_1 + T_1)$$

$$+ \frac{1.1302b_1}{2}(T_2 + T_0) + \frac{0.2715b_1}{2}(T_3 + T_1) + \frac{0.0443b_1}{2}(T_4 + T_2)$$

$$a_0 = 1.2661 + \frac{1.1302b_1}{2} \qquad a_1 = 1.1302 + 1.2661b_1 + \frac{0.2715b_1}{2}$$

$$a_2 = 0.2715 + \frac{1.1302b_1}{2} + \frac{0.0443b_1}{2} \qquad 0 = 0.0443 + \frac{0.2715b_1}{2}$$

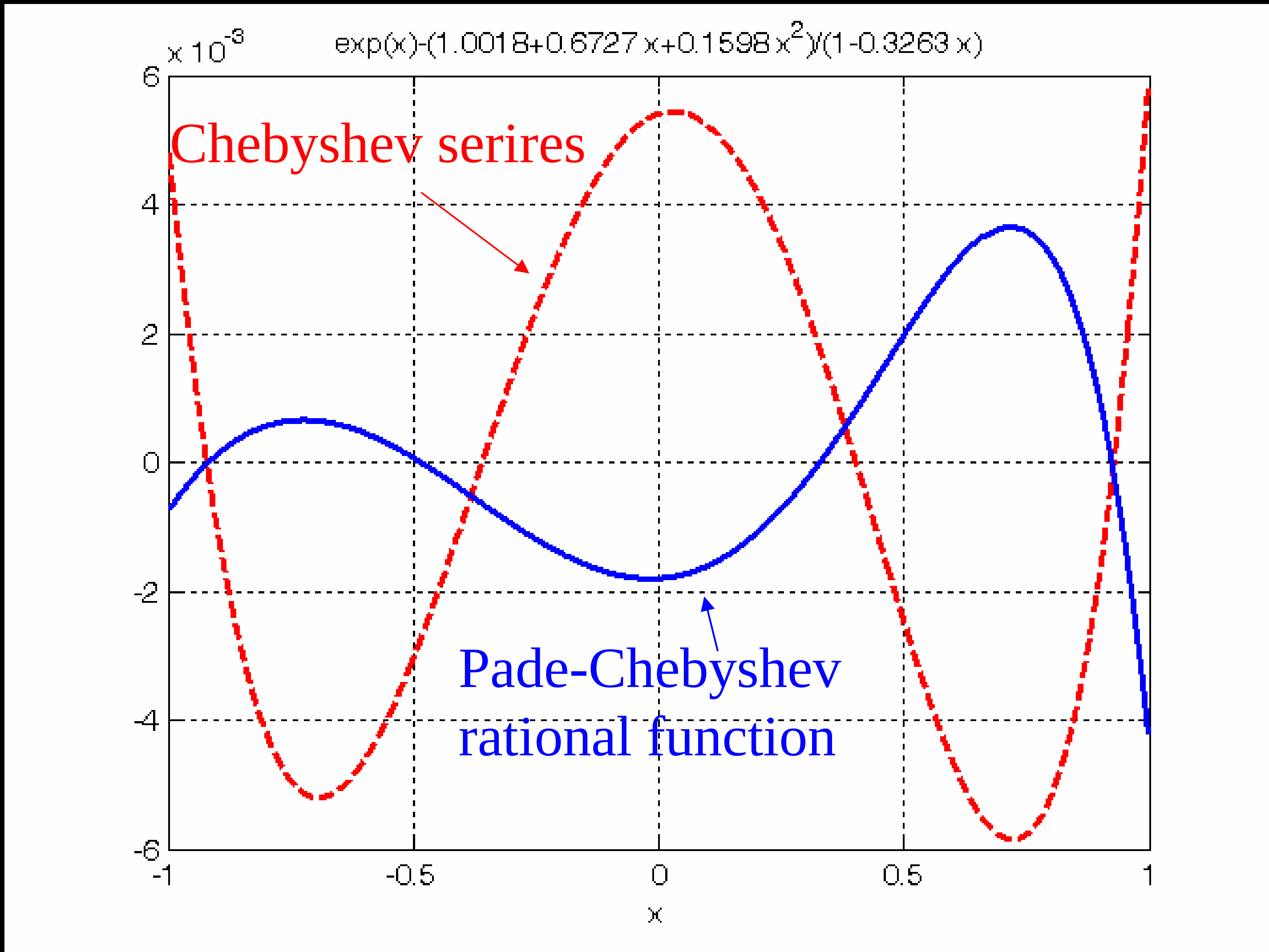
$$e^x \approx \frac{1.0817 + 0.6727T_1 + 0.0799T_2}{1 - 0.3263T_1} = \frac{1.0018 + 0.6727x + 0.1598x^2}{1 - 0.3263x}$$

Chebyshev vs. Chebyshev-Padé Rational

Table 4.5 Comparison of rational approximations [Eq. (4.13)] with Chebyshev series for e^x

x	e^x	Chebyshev	Error	Rational function	Error
−1.0	0.3679	0.3631	0.0048	0.3686	−0.0007
−0.8	0.4493	0.4536	−0.0042	0.4488	0.0006
−0.6	0.5488	0.5534	−0.0046	0.5484	0.0005
−0.4	0.6703	0.6712	−0.0009	0.6707	−0.0004
−0.2	0.8187	0.8154	0.0033	0.8201	−0.0014
0	1.0000	0.9946	0.0054	1.0018	−0.0018
0.2	1.2214	1.2172	0.0042	1.2225	−0.0011
0.4	1.4918	1.4917	0.0001	1.4911	0.0008
0.6	1.8221	1.8267	−0.0046	1.8191	0.0030
0.8	2.2255	2.2307	−0.0051	2.2224	0.0032
1.0	2.7183	2.7121	0.0062	2.7227	−0.0044

Error Plot


x 10^-3
exp(x)-(1.0018+0.6727 x+0.1598 x^2)/(1-0.3263 x)
6
4
2
0
-2
-4
-6
-1
-0.5
0
0.5
1
x
Chebyshev serires
Pade-Chebyshev
rational function

Fourier Series

- Representing a function as a trigonometric series

$$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(nx) + B_n \sin(nx) \right]$$

- $f(x)$ need be integrable so that the coefficients can be computed

Properties of Orthogonality

$$\int_{-\pi}^{\pi} \sin(nx)dx = 0 \qquad \int_{-\pi}^{\pi} \cos(nx)dx = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

Computing Fourier Coefficients using Orthogonality

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(nx) + B_n \sin(nx)\right]$$

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2} dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} A_n \cos(nx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} B_n \sin(nx)dx$$

$$= A_0\pi + 0 + 0$$

$$\int_{-\pi}^{\pi} \cos(mx) f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2} \cos(mx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} A_n \cos(nx)\cos(mx)dx$$

$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} B_n \sin(nx)\cos(mx)dx$$

$$= 0 + A_m\pi + 0$$

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx) f(x)dx, \qquad m = 1,2\ldots$$

Computing Fourier Coefficients using Orthogonality (cont.)

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(nx) + B_n \sin(nx) \right]$$

$$\int_{-\pi}^{\pi} \sin(mx) f(x) dx = \int_{-\pi}^{\pi} \frac{A_0}{2} \sin(mx) dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} A_n \cos(nx) \sin(mx) dx$$

$$+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} B_n \sin(nx) \sin(mx) dx$$

$$= 0 + B_m \pi + 0$$

$$B_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) dx, \qquad m = 1, 2 \ldots.$$

Fourier Series for Periods other than 2 π

- Let the period of $f(x)$ be P
- Consider that $f(x)$ is periodic in $[-P/2, P/2]$
- Change of variable $y = \frac{x\pi}{P/2}$

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx) f(x) dx, \qquad m = 1,2....$$

$$\longrightarrow \quad A_m = \frac{2}{P}\int_{-P/2}^{P/2} f(x) \cos(\frac{2m\pi x}{P}) dx, \qquad m = 1,2....$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(mx) dx, \qquad m = 1,2....$$

$$\longrightarrow \quad B_m = \frac{2}{P}\int_{-P/2}^{P/2} f(x) \sin(\frac{2m\pi x}{P}) dx, \qquad m = 1,2....$$

Fourier Series for Periods other than 2 π

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(\frac{2n\pi x}{P}) + B_n \sin(\frac{2n\pi x}{P}) \right]$$

$$A_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \cos(\frac{2n\pi x}{P}) dx, \qquad n = 1,2....$$

$$B_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \sin(\frac{2n\pi x}{P}) dx, \qquad n = 1,2....$$

Example: $f(x) = x$, x in $[-\pi, \pi]$

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx) f(x) dx, \qquad m = 1,2....$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx) x dx = \frac{1}{\pi}\left(x\frac{\sin(mx)}{m} - \int \frac{\sin(mx)}{m} dx \right)_{-\pi}^{\pi} = 0$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(mx) dx, \qquad m = 1,2....$$

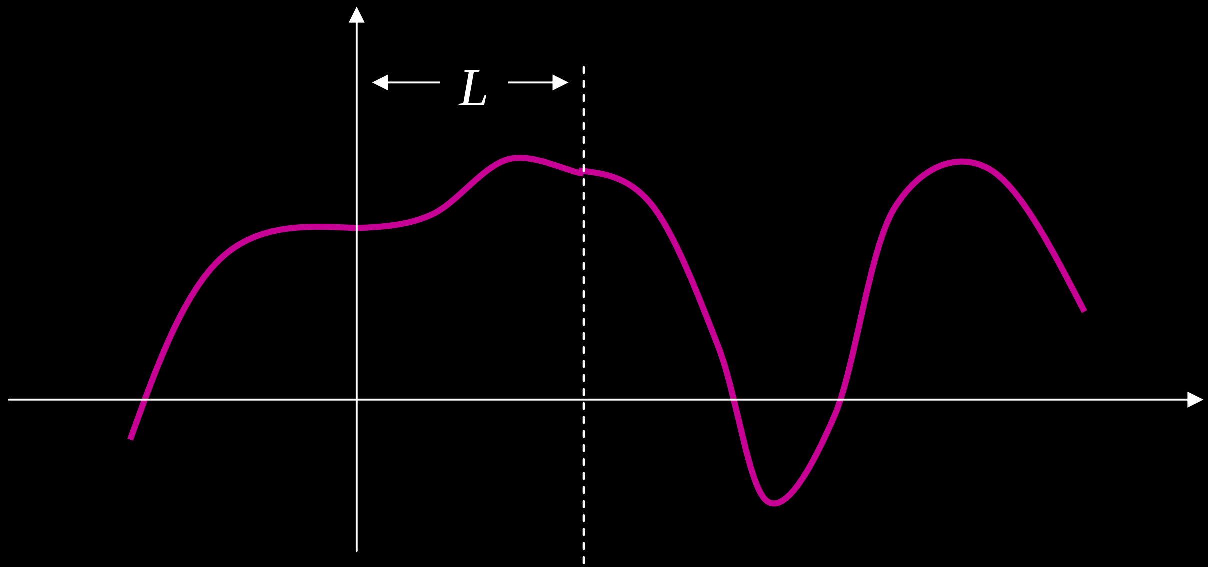
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx) x dx = \frac{1}{\pi}\left(x\frac{-\cos(mx)}{m} - \int \frac{-\cos(mx)}{m} dx \right)_{-\pi}^{\pi}$$

$$= \frac{-2\cos(m\pi)}{m} = \frac{2(-1)^{m+1}}{m}$$

$$x \approx \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{m} \sin(mx)$$

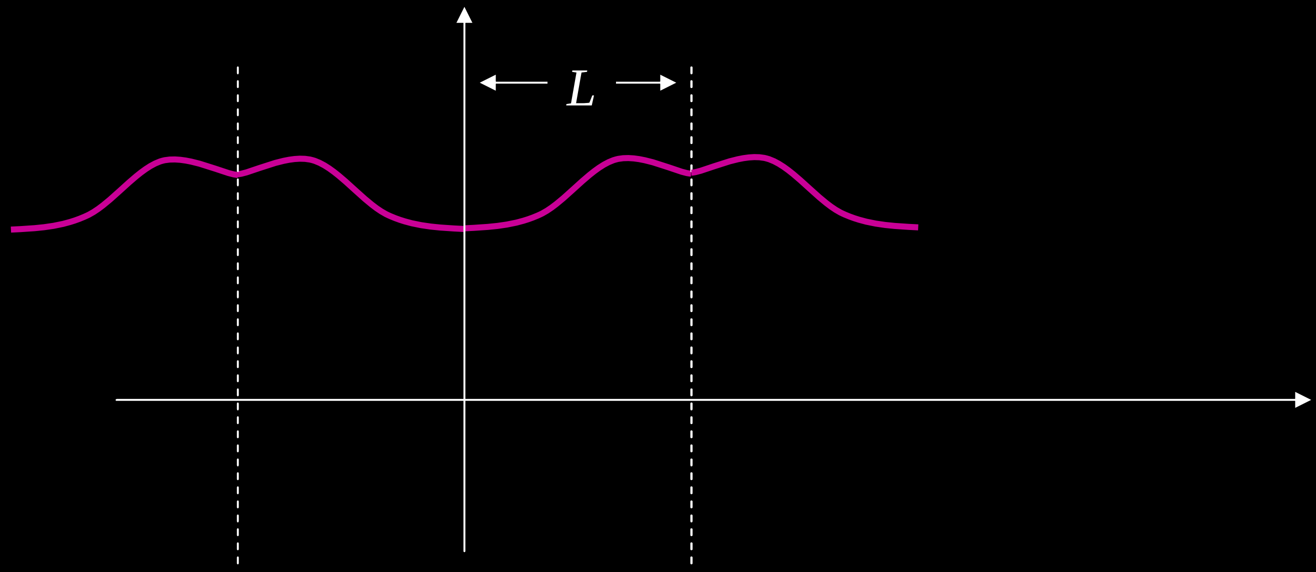
Fourier Series for Nonperiodic Functions

- Chop the interval of interest [0, L]



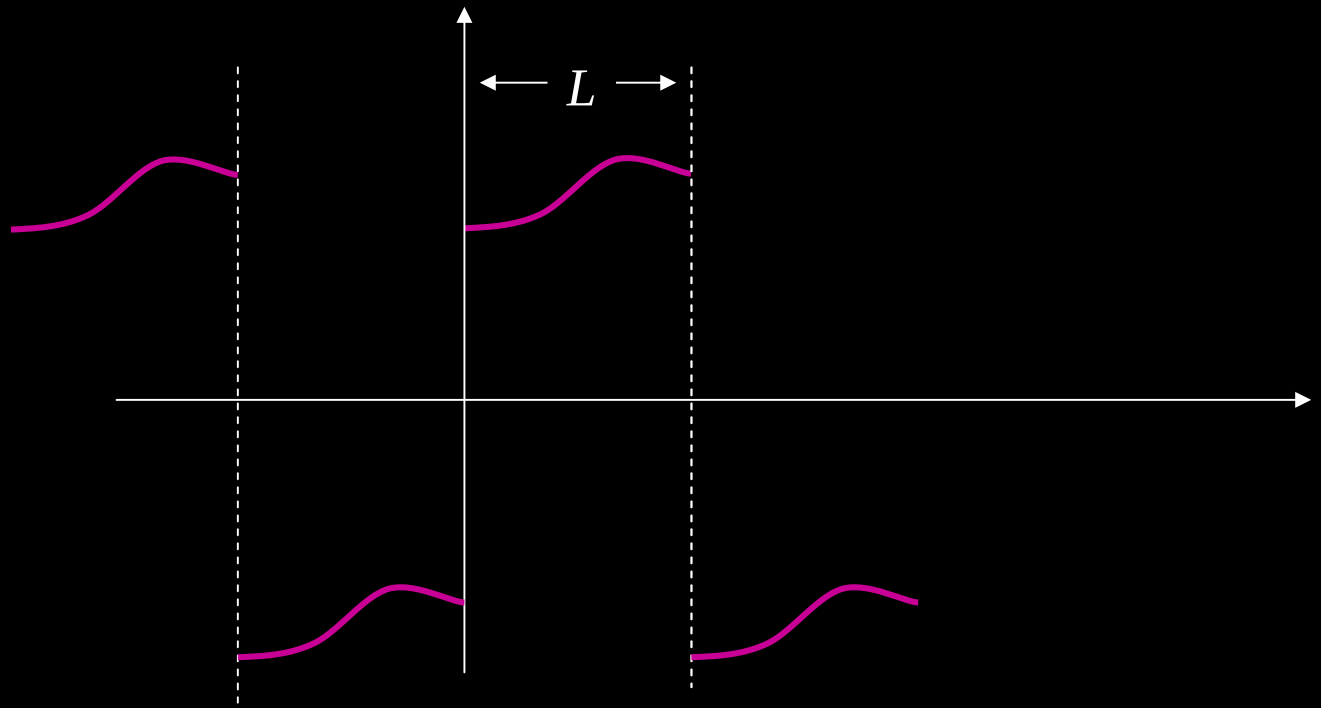
Fourier Series for Nonperiodic Functions

- Solution 1: make an even function by reflecting the segment about y-axis, $f(x)=f(-x)$



Fourier Series for Nonperiodic Functions

- Solution 2: make an odd function by reflecting the segment about the origin, $f(x) = -f(x)$



Properties of Even and Odd Functions

- Product of two even functions is an even function
- Product of two odd functions is an even function
- Product of an even function and an odd function is an odd function

$$\text{if } f(x) \text{ is even} \quad \int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

$$\text{if } f(x) \text{ is odd} \quad \int_{-L}^{L} f(x)dx = 0$$

Fourier Series for Even Functions

$$A_m = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos(\frac{2m\pi x}{P})dx, \qquad m = 1,2....$$

$$= \frac{2}{2L}\int_{-L}^{L} \boxed{f(x)\cos(\frac{2m\pi x}{2L})}dx$$

even function

$$= \frac{1}{L}2\int_{0}^{L} f(x)\cos(\frac{m\pi x}{L})dx$$

$$B_m = \frac{2}{P}\int_{-P/2}^{P/2} \boxed{f(x)\sin(\frac{2m\pi x}{P})}dx, \qquad m = 1,2....$$

$$= 0$$

odd function

Fourier Series for Odd Functions

$$A_m = \frac{2}{P}\int_{-P/2}^{P/2} \boxed{f(x)\cos(\frac{2m\pi x}{P})} dx, \qquad m = 1,2....$$

$$= 0$$

odd function

$$B_m = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin(\frac{2m\pi x}{P}) dx, \qquad m = 1,2....$$

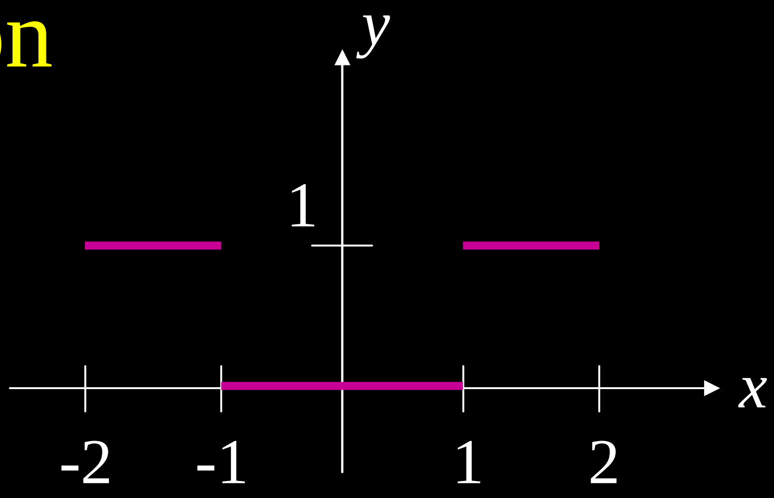
$$= \frac{2}{2L}\int_{-L}^{L} \boxed{f(x)\sin(\frac{2m\pi x}{2L})} dx$$

even function

$$= \frac{1}{L} 2\int_{0}^{L} f(x)\sin(\frac{m\pi x}{L}) dx$$

Example: Fourier Series of Nonperiodic Functions—even extension

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$$



$$A_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})dx$$

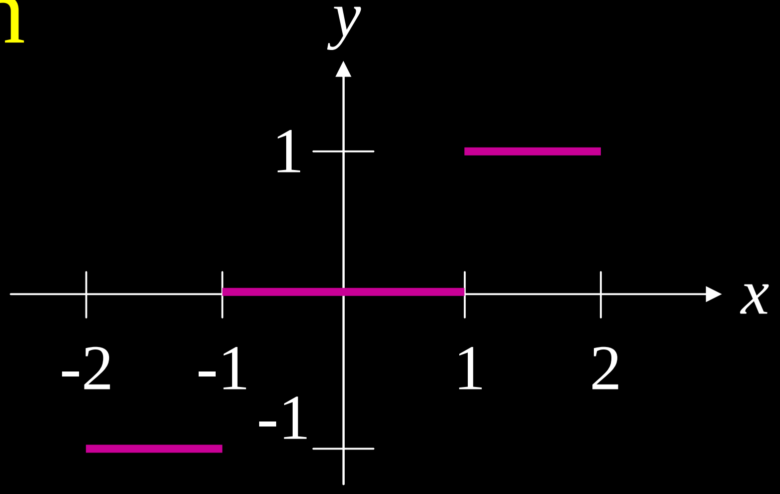
$$= \frac{2}{2}\int_0^2 f(x)\cos(\frac{m\pi x}{2})dx = \begin{cases} 0, & m \text{ even} \\ 2\dfrac{(-1)^{(m+1)/2}}{m\pi}, & m \text{ odd} \end{cases}$$

$$A_0 = \frac{2}{2}\int_0^2 f(x)dx = 1$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n\cos(\frac{2n\pi x}{P}) + B_n\sin(\frac{2n\pi x}{P})\right]$$

Example: Fourier Series of Nonperiodic Functions—odd extension

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$$



$$B_m = \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi x}{L})dx$$

$$= \frac{2}{2}\int_0^2 f(x)\sin(\frac{m\pi x}{2})dx \; = \frac{2}{m\pi}\left[-\cos(m\pi) + \cos(\frac{m\pi}{2})\right]$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(\frac{2n\pi x}{P}) + B_n \sin(\frac{2n\pi x}{P})\right]$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-\cos(n\pi) + \cos(\frac{n\pi}{2})}{n}\sin(\frac{n\pi x}{2})$$