IMAGE-BASED MODELING AND RENDERING

APPENDIX 2. LEAST SQUARES

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Objective

- Least square methods
  - Linear
  - Non-linear

Ref :

• Prof. D.A. Forsyth, Computer Vision, UIUC.
• David A. Forsyth and Jean Ponce, Computer Vision: A Modern Approach, Prentice Hall, New Jersey, 2003.

Linear least-squares methods

- *P* linear equations in *q* unknowns:
- *Ux* = *y*

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1q} \\ u_{21} & u_{22} & \cdots & u_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p1} & u_{p2} & \cdots & u_{pq} \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_q \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_p \end{bmatrix}$$

- When p < q: a (q-p) dimensional vector space
- When p = q: a unique solution
- When p > q: overconstrained system

Normal equations and pseudo-inverse

- min $E = |Ux\text{-}y|^2 = e^T e$, where e = $Ux\text{-}y$.
- The minimum E occurs when the derivatives are zeros.
- We define vector c_j = the j^th column of U

$$\frac{\partial e}{\partial x_i} = \partial\left[\begin{pmatrix} c_1 & \cdots & c_q \end{pmatrix}\begin{pmatrix} x_1 \\ \cdots \\ x_q \end{pmatrix} - y\right] \Big/ \partial x_i = c_i \qquad \frac{\partial E}{\partial x_i} = 2\frac{\partial e}{\partial x_i}\cdot e = 2c_i^T(Ux-y) = 0$$

$$0 = \begin{pmatrix} c_1^T \\ \cdots \\ c_q^T \end{pmatrix}(Ux-y) = U^T(Ux-y) \Leftrightarrow U^T U x = U^T y$$

$$\boxed{x = \left(U^T U\right)^{-1} U^T y}$$

Numerical issue: QR or SVD methods are more reliable

min Ux=0, subject to |x|=1

- Assume y=0, $E=|Ux|^2 = x^T U^T U x$.
- $U^T U$ is symmetric positive semidefinite :
  - $U^T U$'s eigenvalues $0 \le \lambda_1 \le \ldots \le \lambda_q$.
  - $U^T U$ can be decomposed as QDQ^{-1}, where Q and D consist of eigenvectors and eigenvalue respectively.

- Unit vector x can be represented in terms of eigen vector e_i:
  - $x = \mu_1 e_1 + \mu_2 e_2 + \ldots + \mu_q e_q$ and $\mu_1^2 + \ldots. + \mu_q^2 = 1$

- $E(x) - E(e_1) = x^T U^T U x - e_1^T U^T U e_1 = \lambda_1 \mu_1^2 + \ldots. + \lambda_q \mu_q^2 - \lambda_1$

$$\ge \lambda_1(\mu_1^2 + \ldots. + \mu_q^2 - 1) = 0$$

The x that minimize E is eigenvector e_1 of $U^T U$

Nonlinear least-squares methods

- P general equations in q unknowns:

$$f_1(x_1, x_2, \cdots, x_q) = 0$$
$$f_2(x_1, x_2, \cdots, x_q) = 0$$
$$\cdots$$
$$f_p(x_1, x_2, \cdots, x_q) = 0$$

- The error function $E(x) = |f(x)|^2 = \sum (f_i(x))^2$
- Taylor expansion of f_i is

$$f_i(x + \delta x) = f_i(x) + \delta x_1 \frac{\partial f_i}{\partial x_1}(x) + \cdots + \delta x_q \frac{\partial f_i}{\partial x_q}(x) + O\left((\delta x)^2\right)$$

$$\approx f_i(x) + \nabla f_i(x) \cdot \delta x$$

Nonlinear least-squares methods (cont.)

$$f(x+\delta x) \approx f(x) + \Im_{f(x)} \delta x$$

$$\Im_{f(x)} = \begin{pmatrix} \nabla f_1^T(x) \\ \cdots \\ \nabla f_p^T(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_q}(x) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_p}{\partial x_1}(x) & \cdots & \frac{\partial f_p}{\partial x_q}(x) \end{pmatrix}$$

Jacobian of f

Newton's method

- For $p=q$ (square system), an iterative algorithm.

- Compute perturbation δx such that $f(x + \delta x) \approx 0$:

$$\Im_{f(x)} \delta x = -f(x)$$

Overconstrained system (p>q)

- Gaussian-Newton method, similar to the pseudo-inverse:

$$\Im_{f(x)}{}^{T}\Im_{f(x)}\delta x = -\Im_{f(x)}{}^{T} f(x)$$

- Levenberg-Marquardt method, to avoid degenerate pseudo-inverse of the Jacobian matrix J_f.

$$(\Im_{f(x)}{}^{T}\Im_{f(x)} + \mu I)\delta x = -\Im_{f(x)}{}^{T} f(x)$$