IMAGE-BASED MODELING AND RENDERING

APPENDIX 1. CONJUGATE GRADIENT

I-Chen Lin, Dept. of CS, National Chiao Tung University

Conjugate Gradient Method

- Minimizing quadratic functions.
  - Also applicable for min $|Ax - b|$.
- An iterative method using low memory requirement.
  - Storing only the update of vectors (instead of matrices).
- The convergence speed is faster than the steepest decent method (according to the gradient direction).

Minimizing a Quadratic Function

- For a matrix A, which is symmetric ($A^T = A$) and positive definite ($x^T A x > 0$ for all nonzero $x \in R^n$).

$$\min \ f(x) = \frac{1}{2} x^T A x - b^T x$$

- The optimal solution occurs at

$$\frac{\partial f(x)}{\partial x} = Ax - b = 0$$

- We can denote the residual r :

$$r(x) = b - Ax$$

Steepest Descent Method

- To minimize a quadratic function, $\min f(x) = \frac{1}{2} x^T A x - b^T x$

1. Assign an initial guess vector x_0. Set $k = 0$.
2. Evaluate the search direction $p_k = r_k = b - Ax_k$.
   - (Leave the iterations, when the residual is small)
3. Evaluate the optimal "span" α_k that $\min_\alpha f(x_k + \alpha_k p_k)$

   $$\alpha_k = \frac{p_k^T p_k}{p_k^T A p_k}$$
4. Update $x_{k+1} = x_k + \alpha_k p_k$
5. Goto Step 2.

Steepest Descent Method (cont.)

- Deriving α_k:

$$f(x_k + \alpha_k p_k) = \frac{1}{2}(x_k + \alpha_k p_k)^T A(x_k + \alpha_k p_k) - b^T (x_k + \alpha_k p_k)$$

$$= \frac{1}{2}\alpha_k^2 p_k^T A p_k + \alpha_k (x_k{}^T A p_k - b^T p_k) + \frac{1}{2} x_k{}^T A x_k - b^T x_k$$

$$\frac{\partial f(x_k + \alpha_k p_k)}{\partial \alpha_k} = \alpha p_k^T A p_k + \left(x_k{}^T A p_k - b^T p_k\right)$$

$$= \alpha_k p_k^T A p_k - p_k^T p_k$$

The optimal α_k $\qquad \alpha_k = \frac{p_k^T p_k}{p_k^T A p_k}$

Conjugate Direction

- A is a symmetric positive definite matrix.
- A-inner-produce $\langle u, v \rangle_A = u^T A v$
- Two non-zero vectors u and v are conjugate (with respect to A) if $u^T A v = 0$.
- Steepest descent method moves in the direction r_k.
- Conjugate gradient method moves along p_k conjugate to each other.

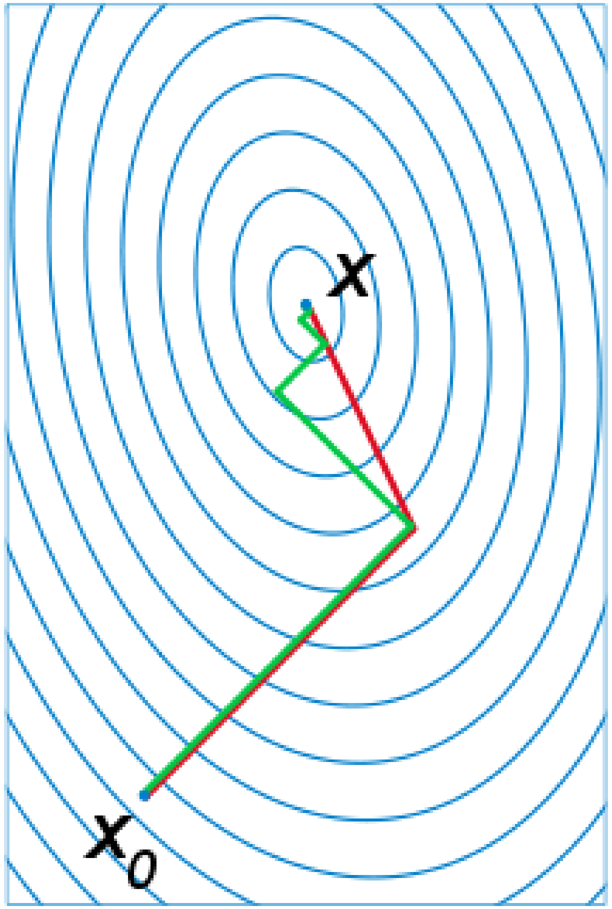

Fig. from: http://en.wikipedia.org/wiki/Conjugate_gradient_method

Conjugate Gradient Method

$$\min \; f(x) = \frac{1}{2} x^T A x - b^T x$$

1. Assign an initial guess vector x_0.
   Set $k = 0$, $r_0 = b - Ax_0$, $p_0 = r_0$.

2. Evaluate the optimal "span" α_k that $\min_{\alpha} \; f(x_k + \alpha_k p_k)$

$$\alpha_k = \frac{r_k^T p_k}{p_k^T A p_k} \qquad x_{k+1} = x_k + \alpha_k p_k \qquad r_{k+1} = r_k - \alpha_k A p_k$$

   (Leave the iterations, when the residual is small)

$$\beta = \frac{-r_{k+1}^T A p_k}{p_k^T A p_k} \qquad p_{k+1} = r_{k+1} + \beta_k p_k$$

3. Goto Step 2.

Conjugate Gradient Method (cont.)

- Derivate α_k: $$\frac{\partial f(x_k + \alpha_k p_k)}{\partial \alpha_k} = \alpha p_k^T A p_k + \left(x_k{}^T A p_k - b^T p_k\right)$$

$$= \alpha_k p_k^T A p_k - r_k^T p_k$$

$$\alpha_k = \frac{r_k^T p_k}{p_k^T A p_k}$$

Due to the orthogonal properties between *p*, *r*

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A p_k}$$

- Let p_{k+1} combined with previous ones:
- Let p_{k+1} be A-conjugate to p_k. $p_{k+1}^T A p_k = 0$ $\quad p_{k+1} = r_{k+1} + \beta_k p_k$
- Derivate β_k: $$p_{k+1}^T A p_k = \left(r_{k+1} + \beta_k p_k\right)^T A p_k = r_{k+1}^T A p_k + \beta_k p_k^T A p_k = 0$$

$$\beta = \frac{-r_{k+1}^T A p_k}{p_k^T A p_k}$$

Conjugate Gradient Method (cont.)

- Derivate β_k (cont.): $\beta = \frac{-r_{k+1}^T A p_k}{p_k^T A p_k}$

Since $r_{k+1} = r_k - \alpha_k A p_k$ $\quad A p_k = \frac{r_k - r_{k+1}}{\alpha_k}$ $\quad r_k \perp r_{k+1}$

$$\beta = \frac{r_{k+1}^T r_{k+1}}{p_k^T A p_k}$$

Since $p_k = r_k + \beta_{k-1} p_{k-1}$ $\quad A p_k = \frac{r_k - r_{k+1}}{\alpha_k}$

The orthogonal properties between *p*, *r*

$$\beta = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$$