Introduction to Computer Graphics

2. Transformations

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Textbook: E.Angel, Interactive Computer Graphics, 5th Ed., Addison Wesley

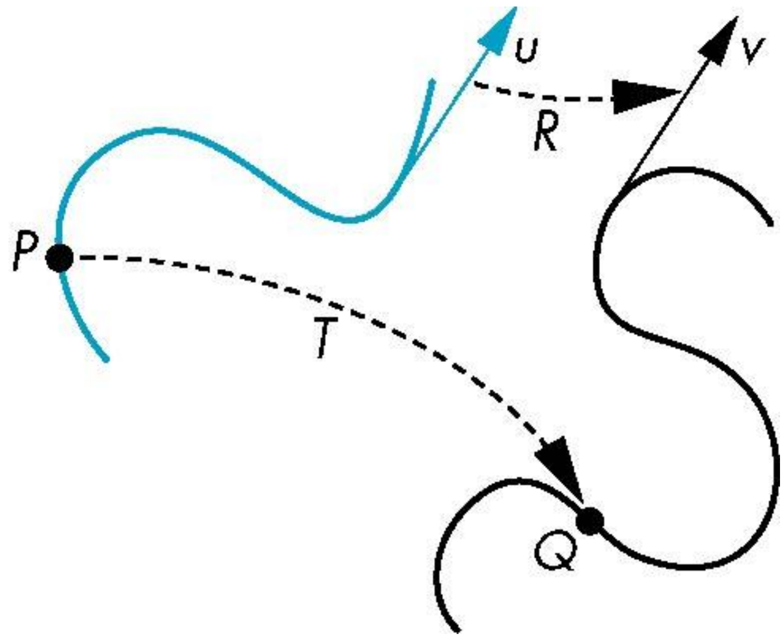
Ref:Hearn and Baker, Computer Graphics, 3rd Ed., Prentice Hall

Outline

- Introduce standard transformations
  - Rotation
  - Translation
  - Scaling
  - Shear
- Derive homogeneous coordinate transformation matrices
- Learn to build arbitrary transformation matrices from simple transformations

General Transformations

- A transformation maps points to other points and/or vectors to other vectors



Affine Transformations

- A transformation that preserves lines and parallelism
  - maps parallel lines to parallel lines
- Characteristic of many physically important transformations
  - Rigid body transformations: rotation, translation
  - Scaling, shear

Translation

- Using the homogeneous coordinate representation in some frame

$$\boldsymbol{p} = [\, x\ y\ z\ 1]^T$$

$$\boldsymbol{p}' = [\, x'\ y'\ z'\ 1]^T$$

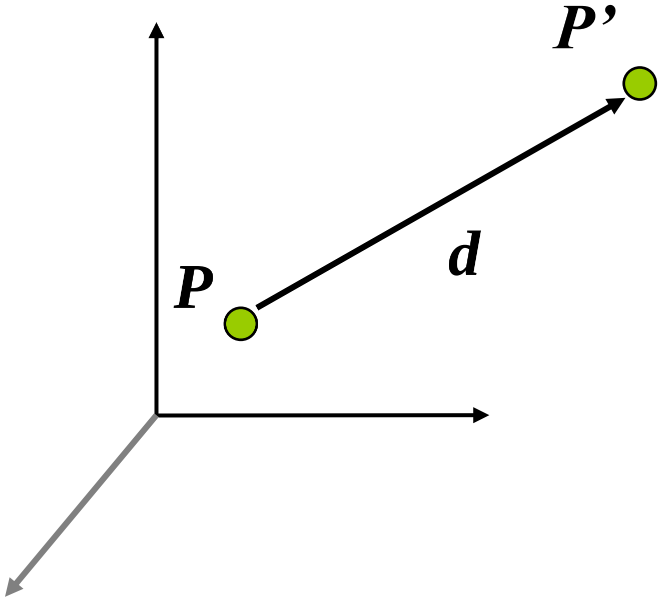
$$\boldsymbol{d} = [\, d_x\ d_y\ d_z\ 0]^T$$

Hence ***p'*** = ***p*** + ***d*** or

$$x' = x + d_x$$

$$y' = y + d_y$$

$$z' = z + d_z$$



Translation Matrix

- We can also express translation using a 4 x 4 matrix **T** in homogeneous coordinates

$$\boldsymbol{p}' = \boldsymbol{T}\boldsymbol{p}$$

where

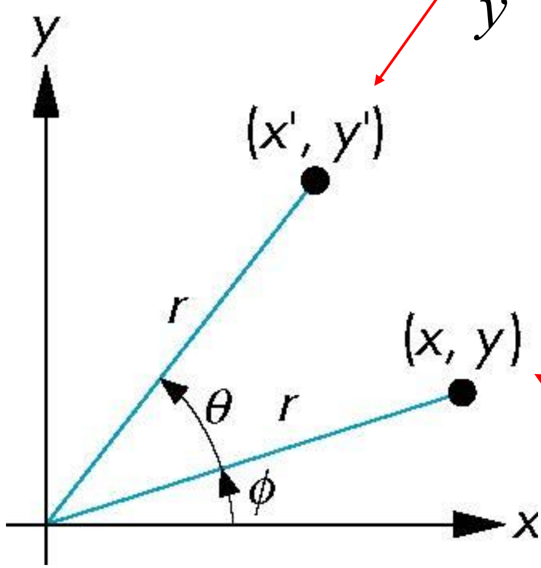
$$\boldsymbol{T} = \boldsymbol{T}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & \mathrm{d}_\mathrm{x} \\ 0 & 1 & 0 & \mathrm{d}_\mathrm{y} \\ 0 & 0 & 1 & \mathrm{d}_\mathrm{z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Why do we use a matrix form instead of vector addition?

Rotation (2D)

- Consider rotation about the origin by q degrees
  - radius stays the same, angle increases by *q*

$$x' = r \cos(\phi + \theta)$$
$$y' = r \sin(\phi + \theta)$$



$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$

$$x = r \cos\phi$$
$$y = r \sin\phi$$

Rotation about the z axis

- Rotation about z axis in three dimensions
  - leaves all points with the same z
  - Equivalent to rotation in two dimensions in planes of constant z

$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$
$$z' = z$$

  - or in homogeneous coordinates

$$\mathbf{p}' = \mathbf{R}_z(\theta)\mathbf{p}$$

Rotation Matrix

$$\boldsymbol{R} = \boldsymbol{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about x and y axes

- Same argument as for rotation about *z* axis
  - For rotation about *x* axis, *x* is unchanged
  - For rotation about *y* axis, *y* is unchanged

$$\mathbf{R} = \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

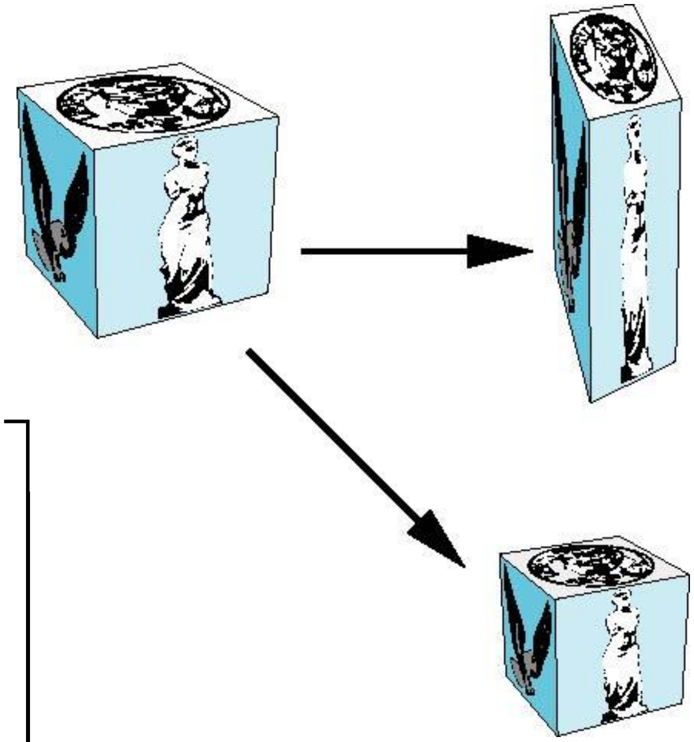
$$\mathbf{R} = \mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- Expand or contract along each axis (fixed point of origin)

$$x' = s_x x$$
$$y' = s_y x$$
$$z' = s_z x$$

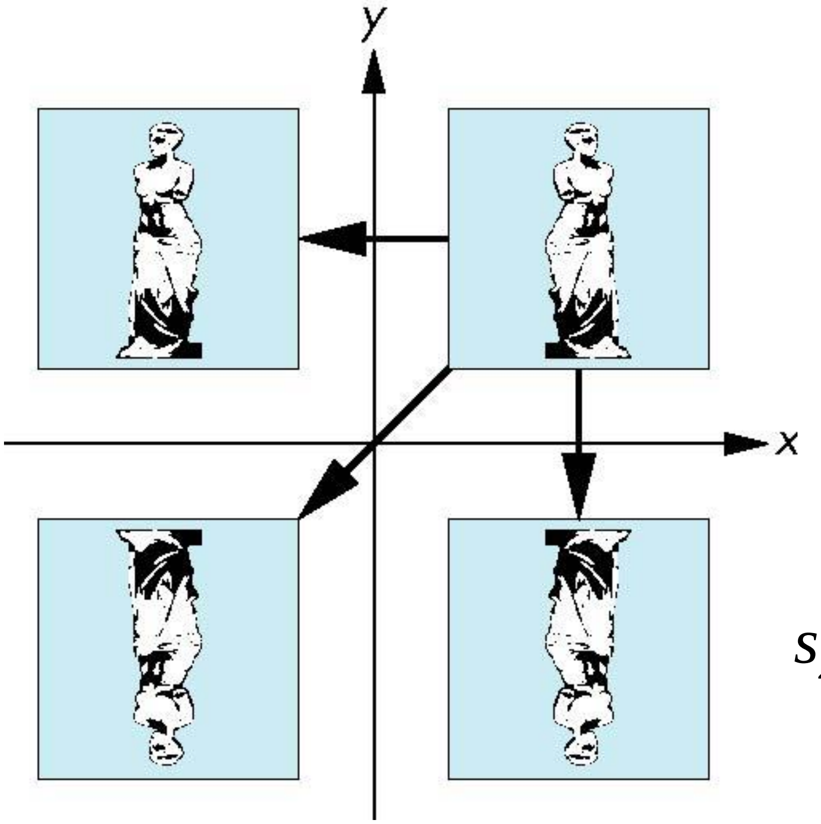
$$\boldsymbol{p}' = \boldsymbol{S}\boldsymbol{p}$$



$$\boldsymbol{S} = \boldsymbol{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection

- corresponds to negative scale factors



$s_x = -1 \; s_y = 1$

original

$s_x = -1 \; s_y = -1$

$s_x = 1 \; s_y = -1$

Inverses

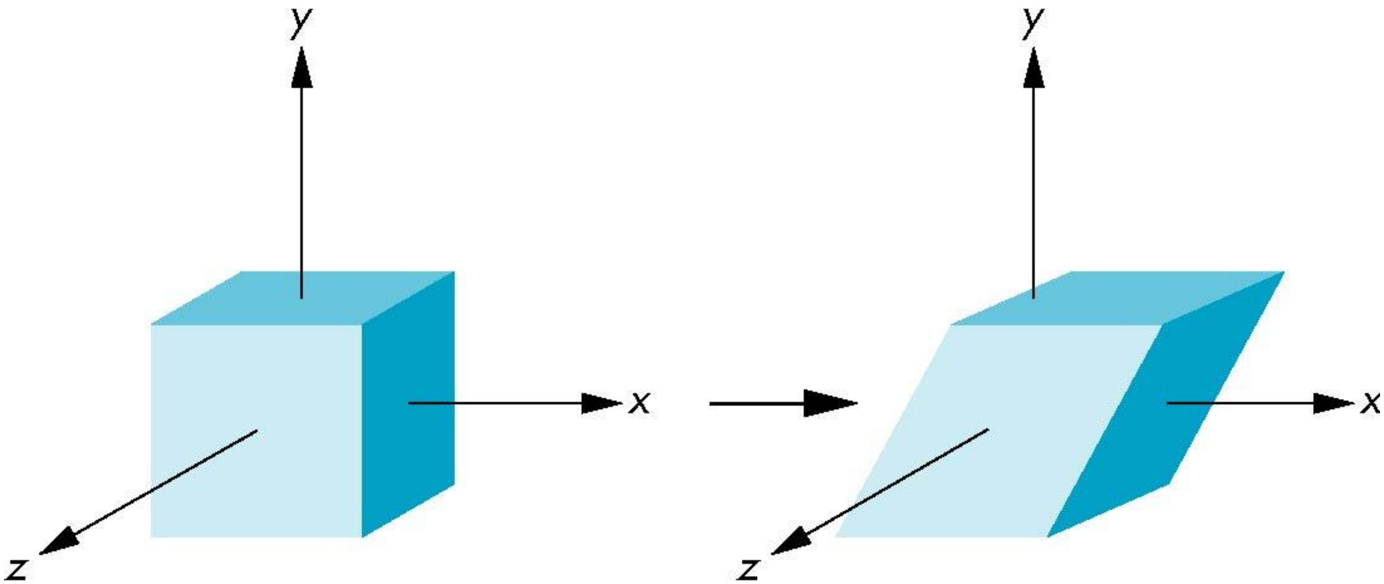
- Compute inverse matrices by general formulas, or use simple geometric observations
  - Translation: $\boldsymbol{T}^{-1}(d_x, d_y, d_z) = \boldsymbol{T}(-d_x, -d_y, -d_z)$
  - Rotation: $\boldsymbol{R}^{-1}(\theta) = \boldsymbol{R}(-\theta)$
    - Holds for any rotation matrix
    - Since $\cos(-\theta) = \cos(\theta)$; $\sin(-\theta) = -\sin(\theta)$

$$\boldsymbol{R}^{-1}(\theta) = \boldsymbol{R}^{T}(\theta)$$

  - Scaling: $\boldsymbol{S}^{-1}(s_x, s_y, s_z) = \boldsymbol{S}(1/s_x, 1/s_y, 1/s_z)$

Shear

- Equivalent to pulling faces in opposite directions



Shear Matrix

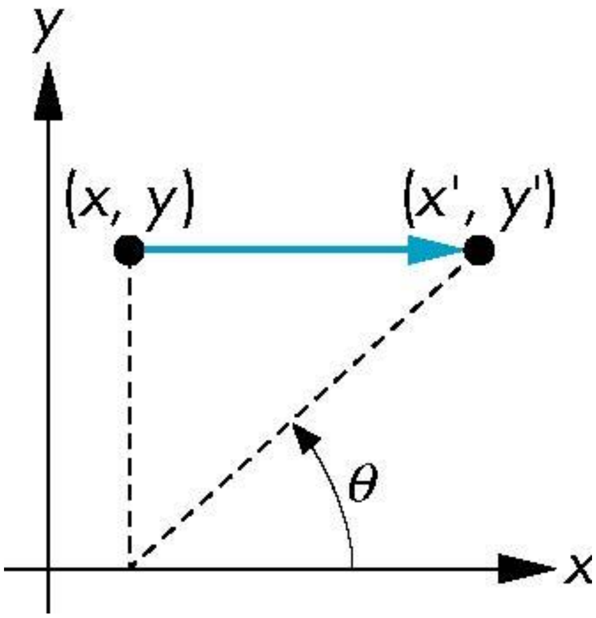
- Consider simple shear along *x* axis

$$x' = x + y \cot\theta$$

$$y' = y$$

$$z' = z$$

$$\mathbf{H}(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Concatenation

- Form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices

$$\textit{for each } i \quad ABCDp_i,$$

or

$$M = ABCD, \quad \textit{for each } i \quad Mp_i$$

Order of Transformations

- Note that matrix on the right is the first applied
- Mathematically, the following are equivalent

$$\mathbf{p}' = \mathbf{ABCp} = \mathbf{A}(\mathbf{B}(\mathbf{Cp}))$$

- Note many references use column matrices to represent points. In terms of column matrices

$$\mathbf{p}'^T = \mathbf{p}^T\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

General Rotation About the Origin

- Decompose into the concatenation of rotations about the *x*, *y*, and *z* axes

$$\boldsymbol{R}(\theta) = \boldsymbol{R}_z(\theta_z)\ \boldsymbol{R}_y(\theta_y)\ \boldsymbol{R}_x(\theta_x)$$

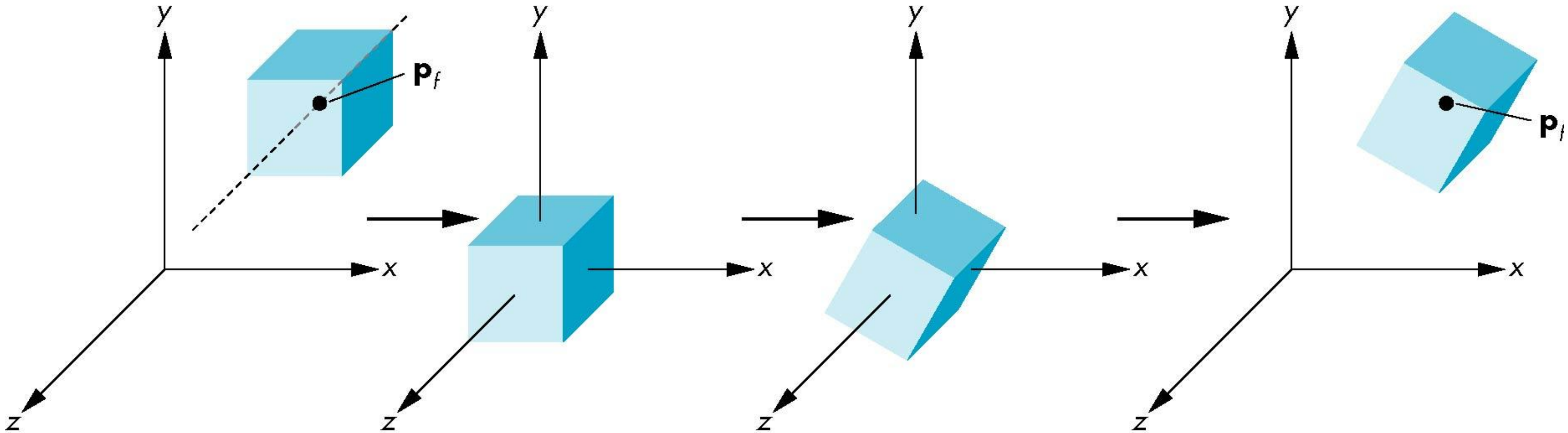
$\theta_x, \theta_y, \theta_z$ are called the Euler angles

- Commutative?

Rotation About a Fixed Point other than the Origin

1. Move fixed point to origin
2. Rotate
3. Move fixed point back

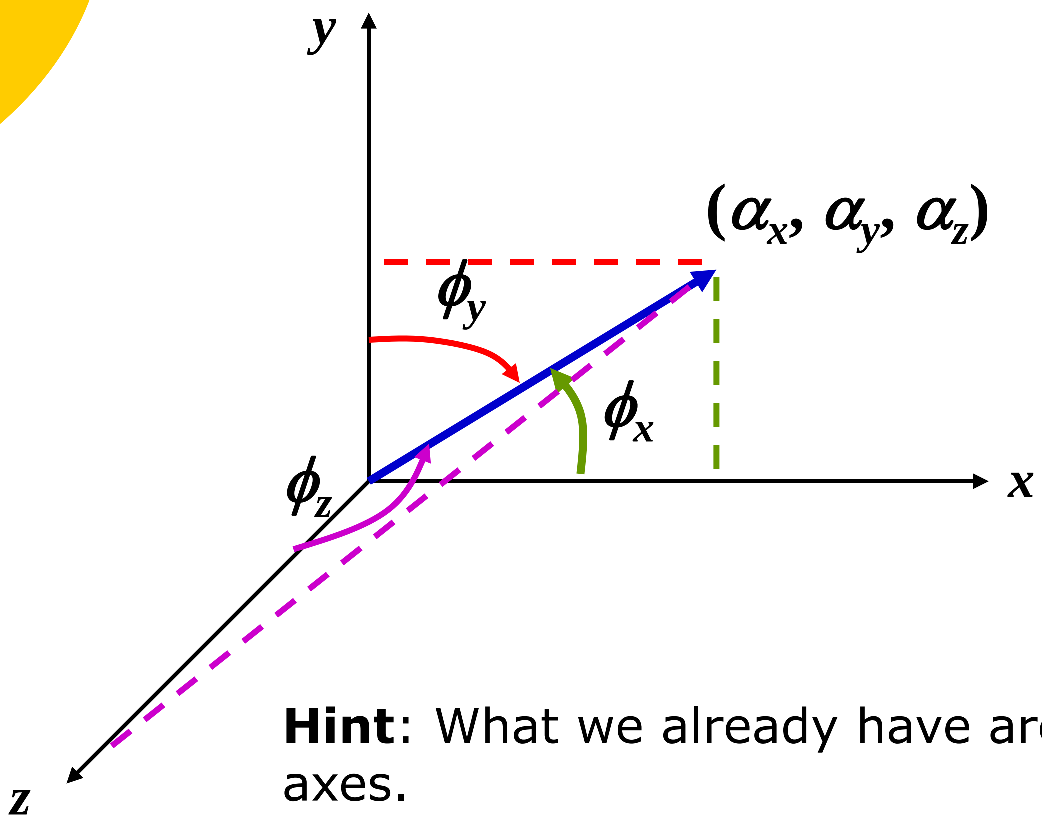
$$\boldsymbol{M} = \boldsymbol{T}(p_f)\,\boldsymbol{R}(\theta)\,\boldsymbol{T}(-p_f)$$



Rotation About an Arbitrary Axis

- Rotate around an axis vector *u*.

$$v = u/|u| = [\alpha_x, \alpha_y, \alpha_z]^T$$



$$\cos\phi_x = \alpha_x$$

$$\cos\phi_y = \alpha_y$$

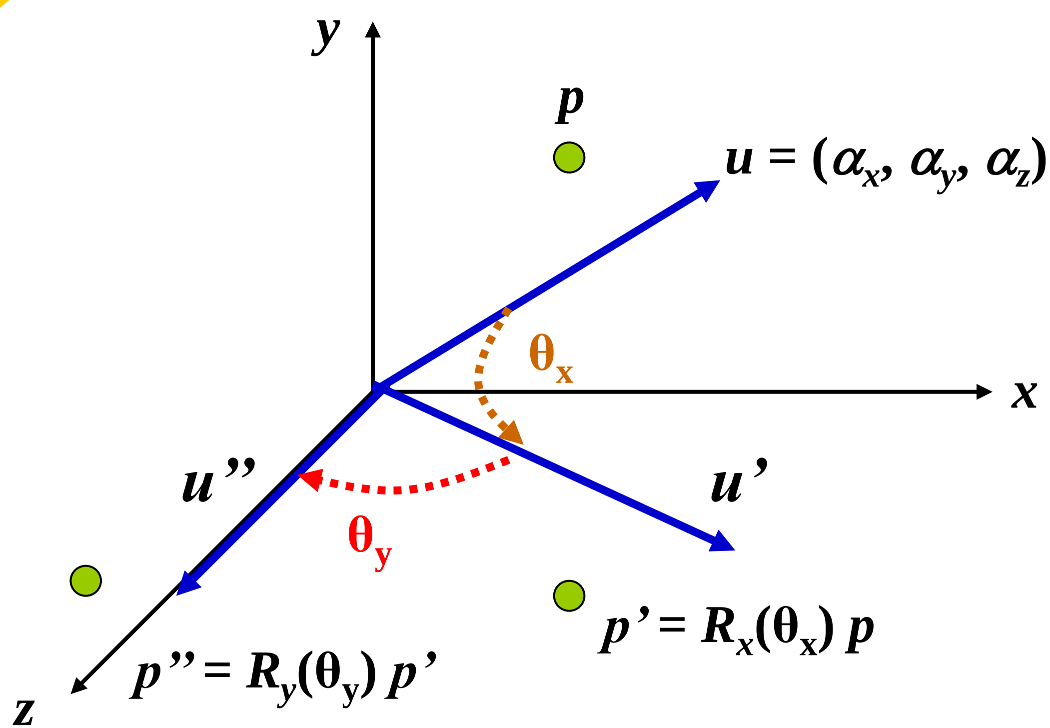
$$\cos\phi_z = \alpha_z$$

$$\cos\phi_x + \cos\phi_y + \cos\phi_z = 1$$

Hint: What we already have are rotations around x, or y, or z axes.

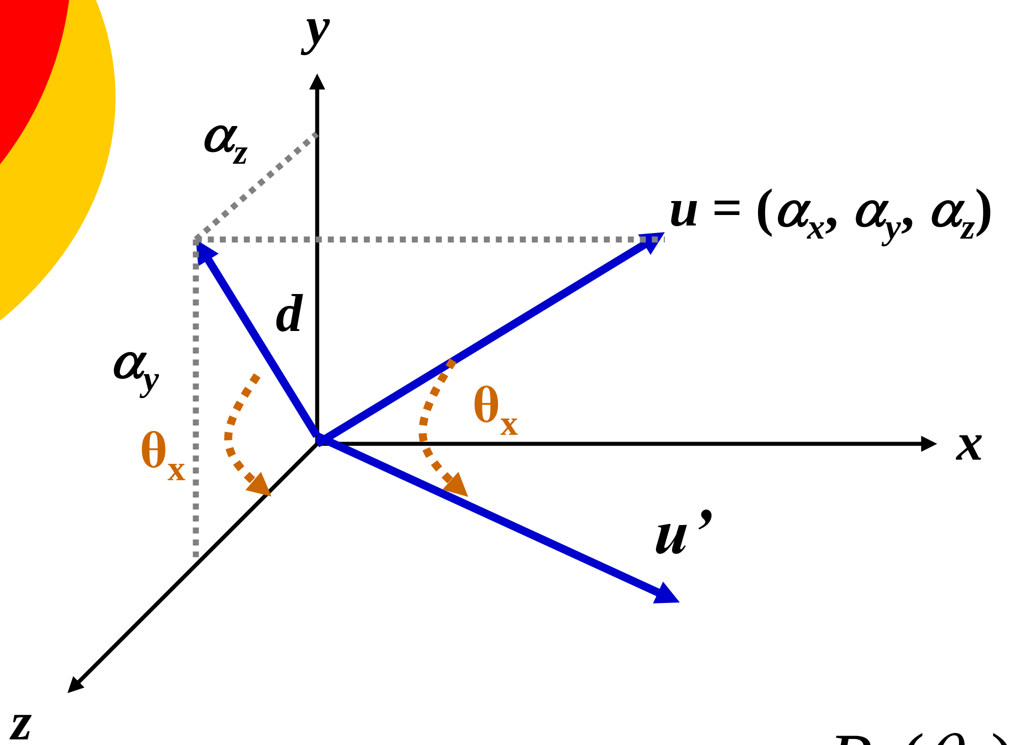
Rotation About an Arbitrary Axis

1. Rotate the axis vector to match *z* (*x* or *y*) axis. [R_{axis}]
2. Rotate around *z* axis. [$R_z(\theta)$]
3. Rotate the axis vector back. [R_{axis}^{-1}]



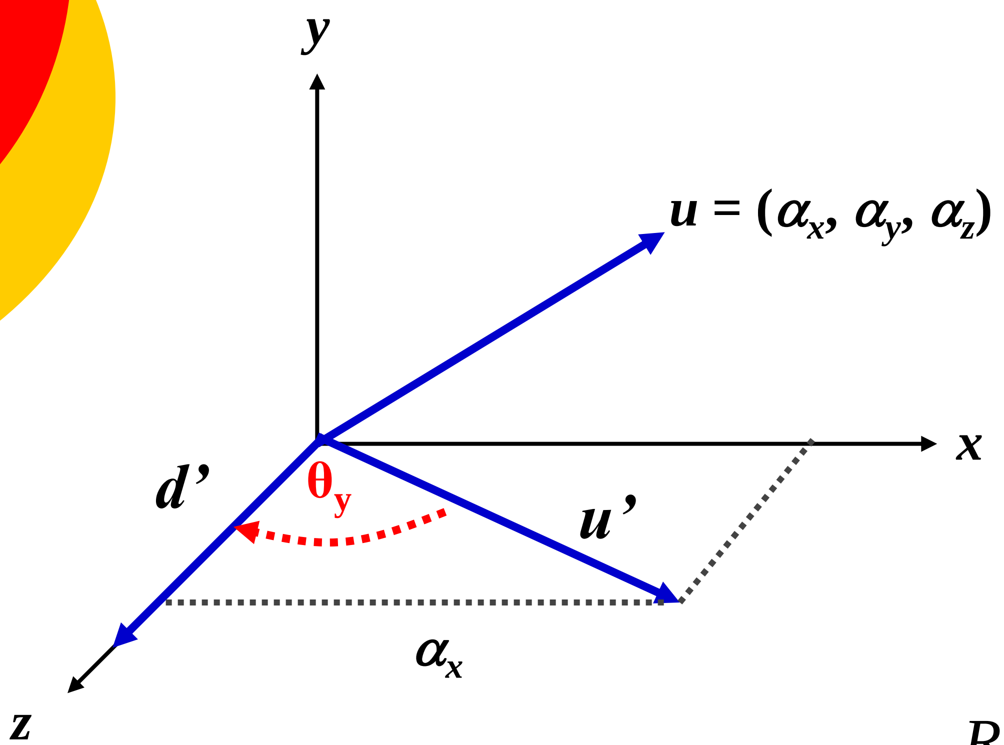
$$R_{axis} = R_y(\theta_y)R_x(\theta_x)$$

$R_x(\theta_x)$



$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_z/d & -\alpha_y/d & 0 \\ 0 & \alpha_y/d & \alpha_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

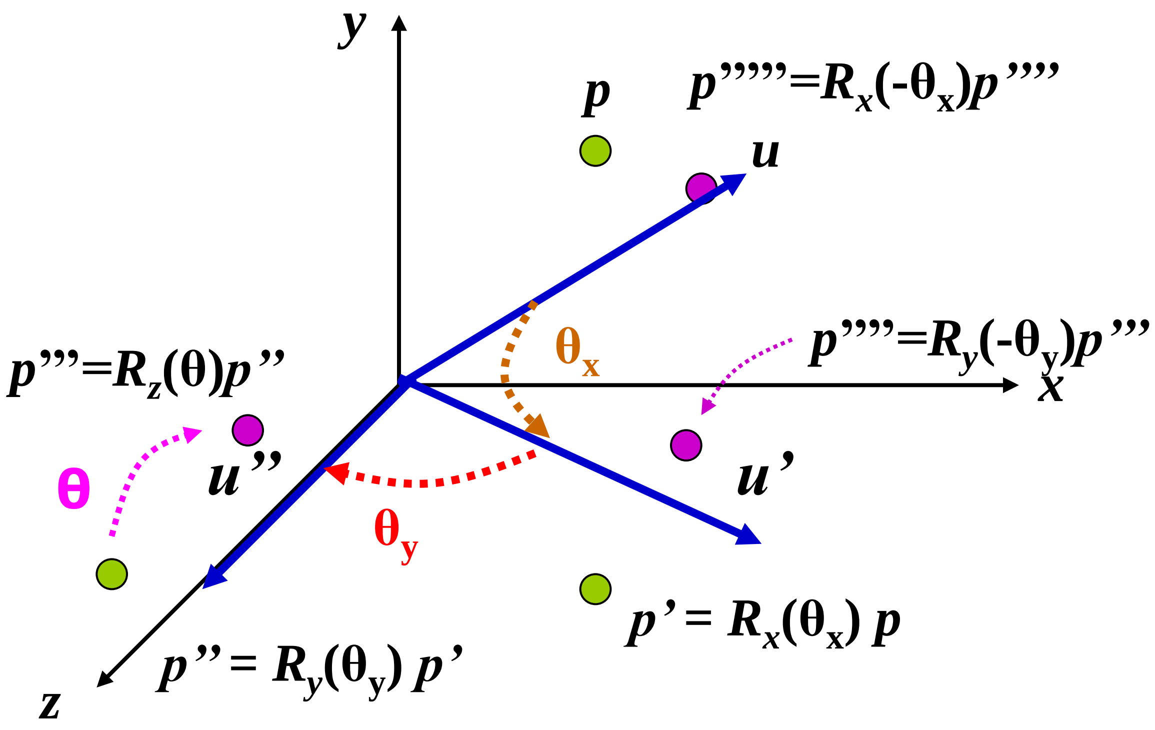
$R_y(\theta_y)$



$$R_y(\theta_y) = \begin{bmatrix} d' & 0 & -\alpha_x & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_x & 0 & d' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

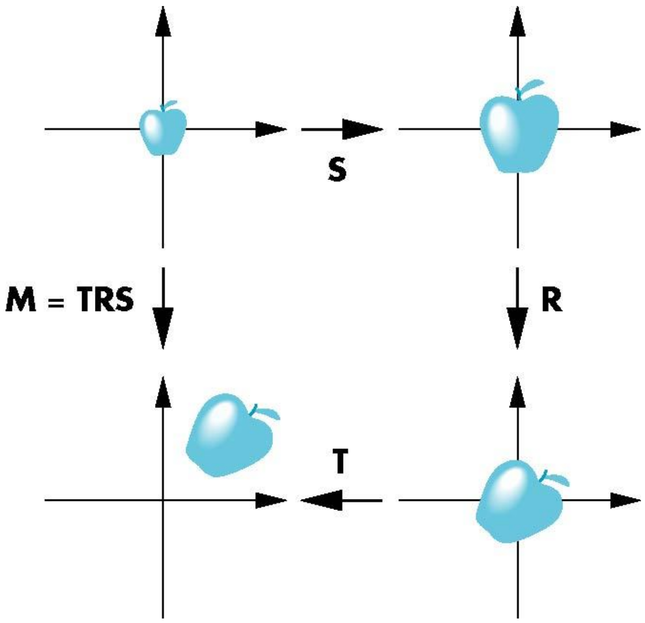
Rotation About an Arbitrary Axis

$$M = R_{axis}^{-1}\, R_z(\theta)\, R_{axis}$$
$$= R_x(-\theta_x)\, R_y(-\theta_y)\, R_z(\theta)\, R_y(\theta_y)\, R_x(\theta_x)$$



Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size
- We apply an *instance transformation* to its vertices to
  - Scale
  - Orient
  - Locate



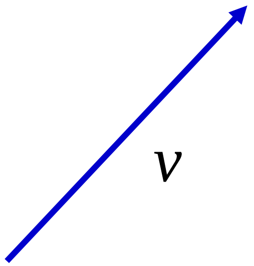
Appendix

Basic Elements

- Geometry:
  - the relationships among objects in an *n-dimensional space*
  - Computer graphics mainly focuses on *three dimensions*.
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalars
  - Vectors
  - Points

Vectors

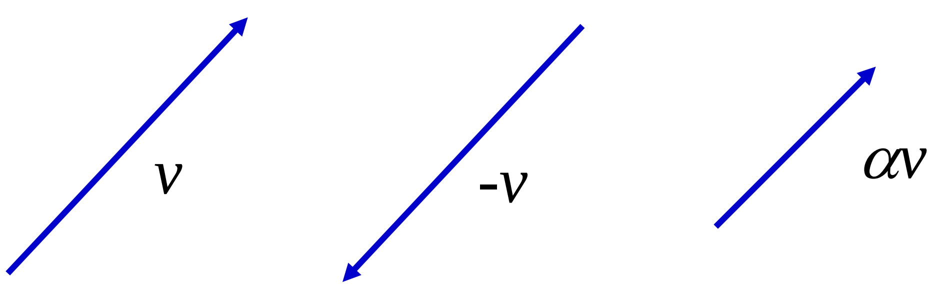
- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude

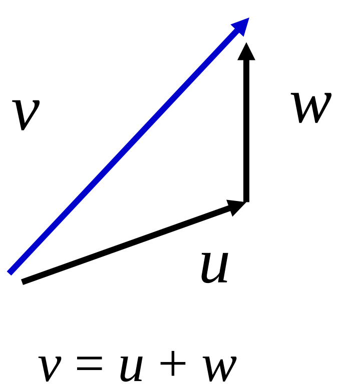


- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types

Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom





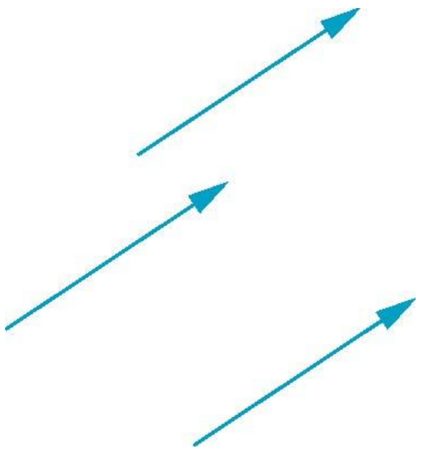
Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication $u=\alpha v$
  - Vector-vector addition: $w=u+v$
- Expressions such as

  $v=u+2w-3r$
- Make sense in a vector space

Vectors Lack Position

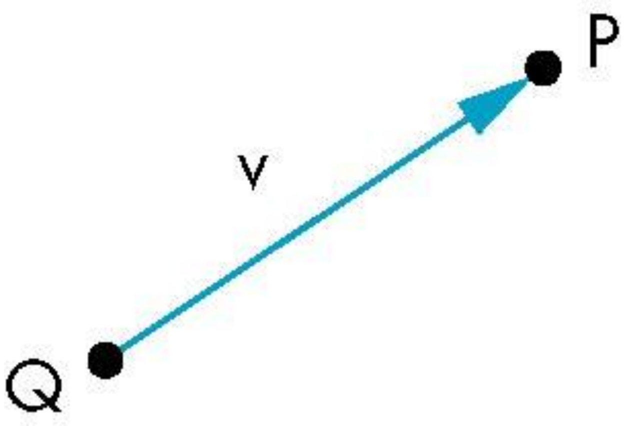
- These vectors are identical
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



$$v = P - Q$$

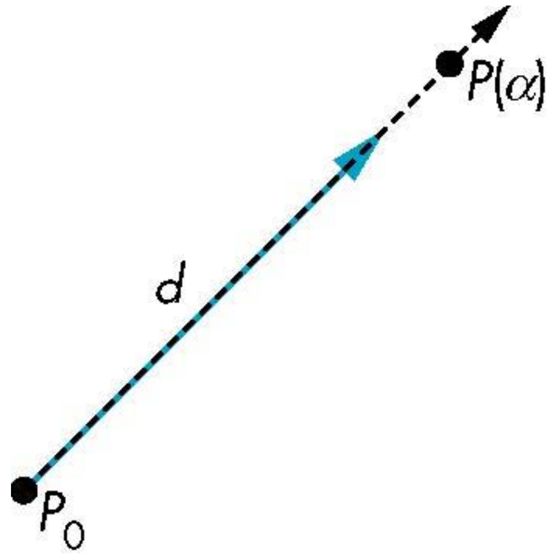
$$P = v + Q$$

Affine Spaces

- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \bullet P = P$
  - $0 \bullet P = \mathbf{0}$ (zero vector)

Lines

- Consider all points of the form
  - $P(\alpha)=P_0 + \alpha \mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector $\mathbf{d}$



Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces

- Two-dimensional forms
  - **Explicit**: $y = mx + h$
  - **Implicit**: $ax + by + c = 0$
  - **Parametric**:

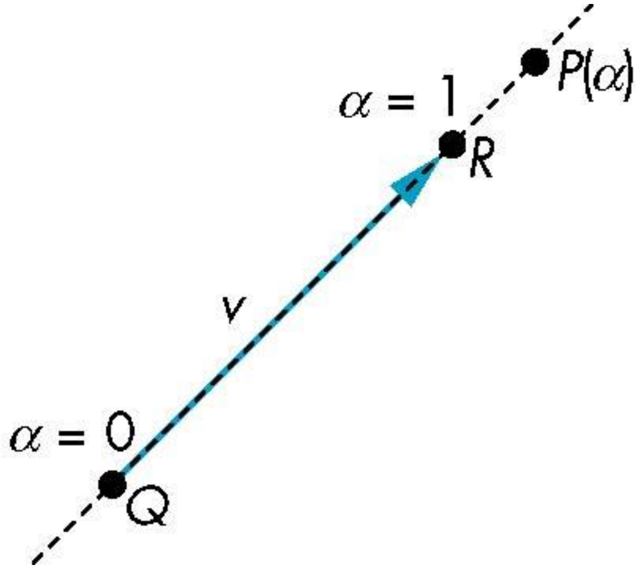
$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$

$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

Rays and Line Segments

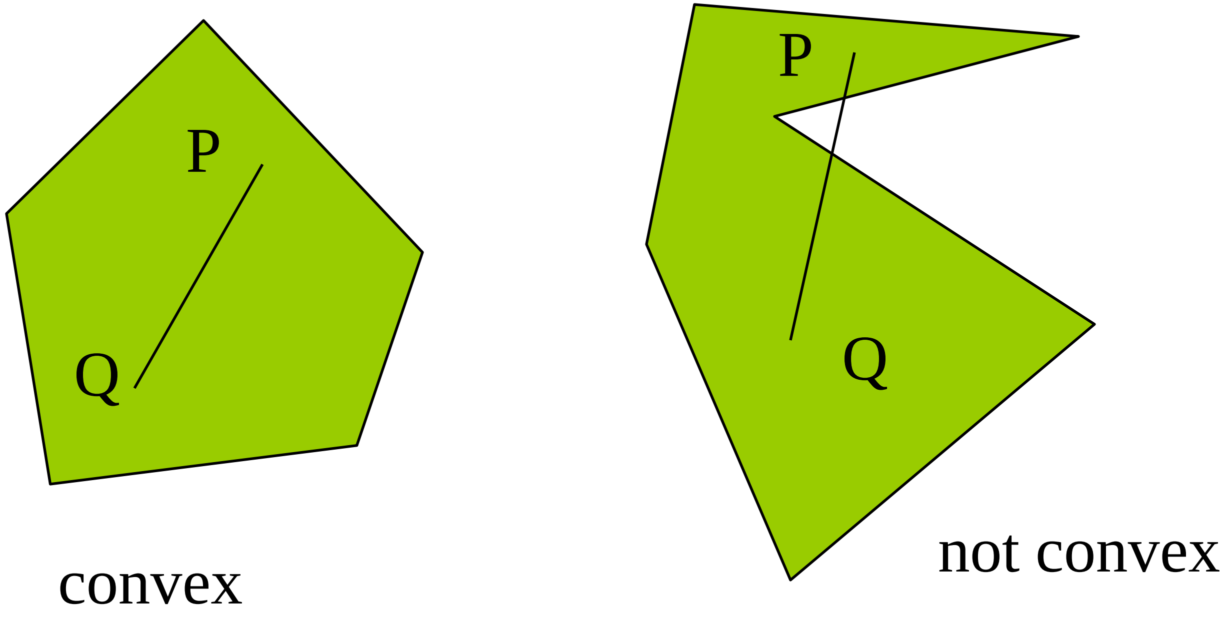
- $\alpha >= 0$, *ray* leaving P_0 in the direction $\mathbf{d}$
- If we use two points to define v, then

  $P(\alpha) = Q + \alpha (R-Q) = Q + \alpha v = \alpha R + (1-\alpha)Q$
- $0 <= \alpha <= 1$, *line segment* joining R and Q



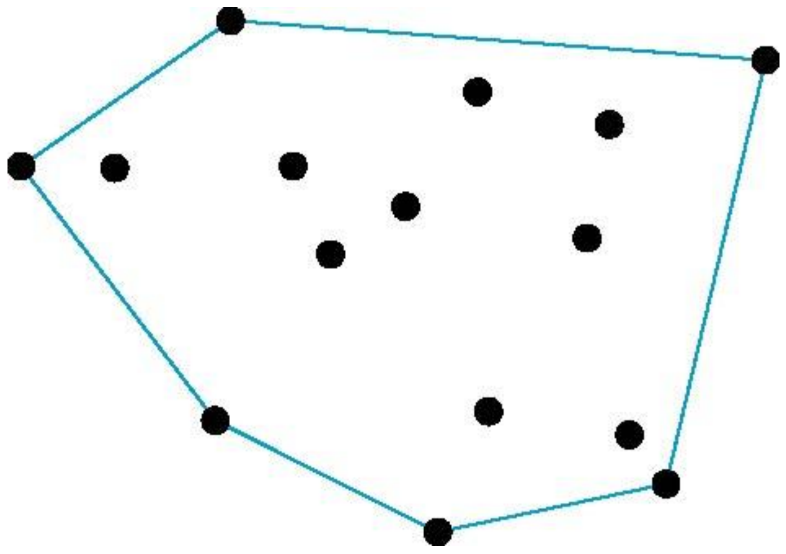
Convexity

- *Convex* iff:
  - for any two points in the object all points on the line segment between these points are also in the object



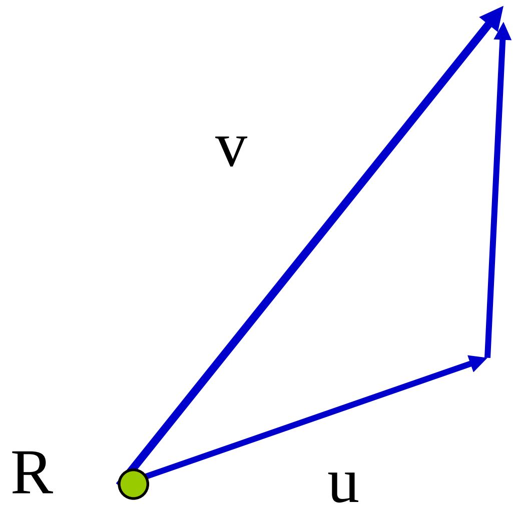
Convex Hull

- Smallest convex object containing $P_1, P_2, \ldots P_n$
- Formed by “shrink wrapping” points

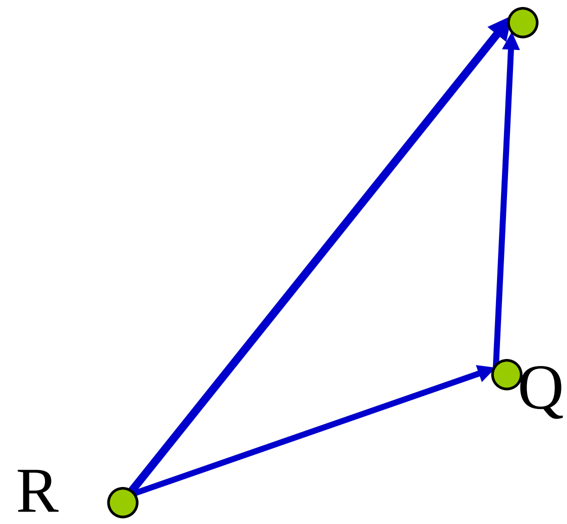


Planes

- A plane can be defined by a point and two vectors or by three points



$P(\alpha,\beta)=R+\alpha u+\beta v$

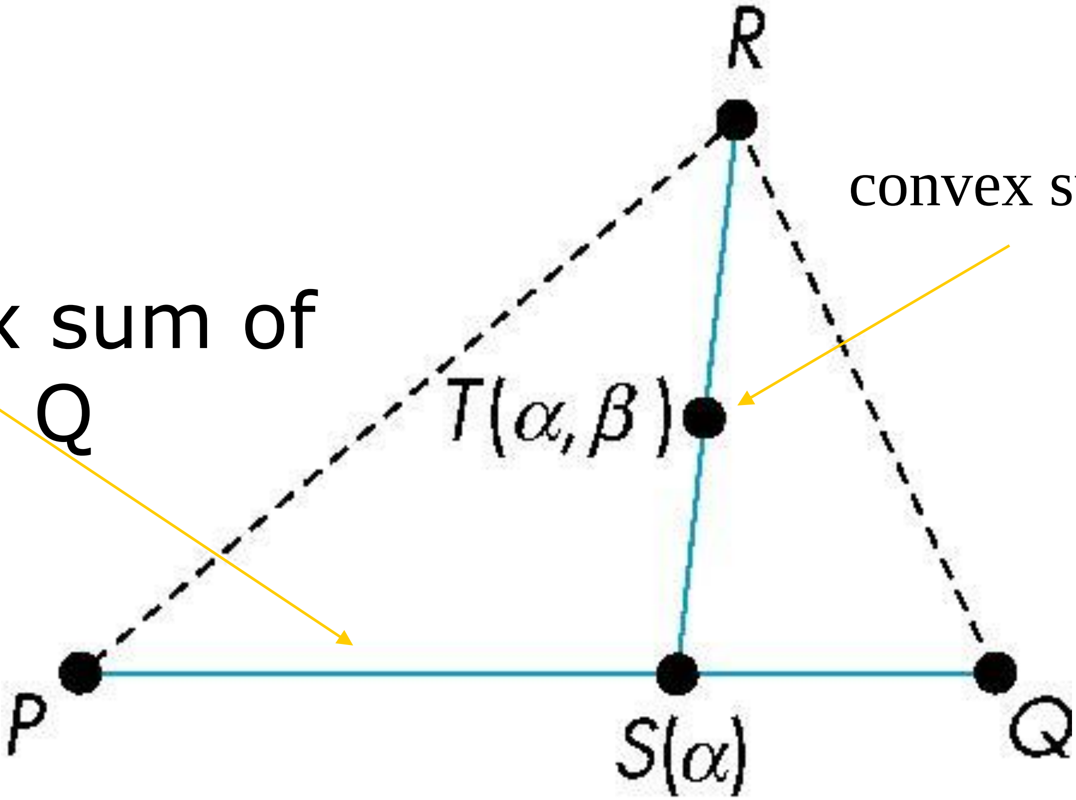


$P(\alpha,\beta)=R+\alpha(Q-R)+\beta(P-Q)$

Triangles

convex sum of $S(\alpha)$ and R

convex sum of P and Q

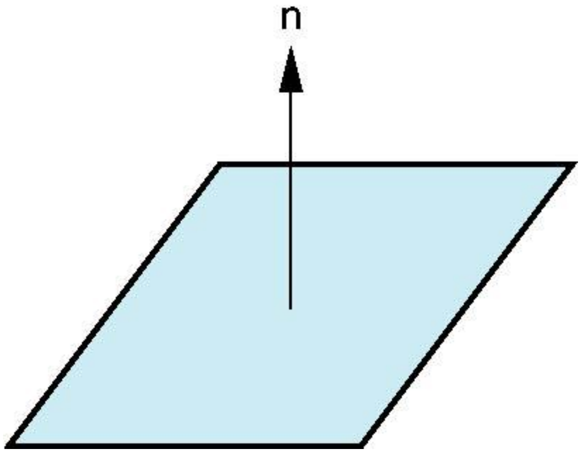


for $0<=\alpha,\beta<=1$, we get all points in triangle

Normals

- Every plane has a vector n normal (perpendicular, orthogonal) to it
- From point-two vector form $P(\alpha,\beta)=R+\alpha u+\beta v$, we know we can use the cross product to find $n = u \times v$ and the equivalent form

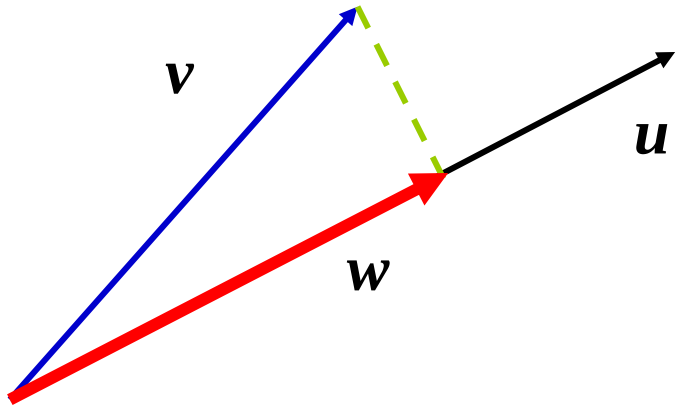
$$(P(\alpha)-P) \cdot n=0$$



Dot product

- $u = [x_1, x_2, x_3]^T$
- $v = [y_1, y_2, y_3]^T$
- $u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3 = |u||v|\cos\theta$

- Projection



$$w = (|v|\cos\theta)\,unit(u)$$

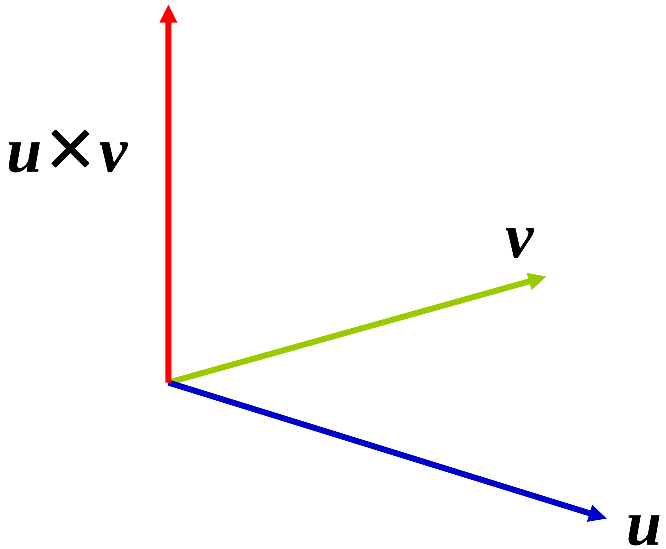
$$= \left(|v|\frac{u \bullet v}{|u||v|}\right)\frac{u}{|u|}$$

$$= \left(\frac{u \bullet v}{|u|^2}\right)u$$

Cross Product

- $u = [x_1, x_2, x_3]^T$
- $v = [y_1, y_2, y_3]^T$
- $| u \times v | = |u||v|\sin\theta|$

$$w = u \times v = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$



Linear Independence

- A set of vectors $v_1, v_2, \ldots, v_n$ is *linearly independent* if

  $$\alpha_1 v_1 + \alpha_2 v_2 + .. \ \alpha_n v_n = 0 \text{ iff } \alpha_1 = \alpha_2 = \ldots = 0$$

- If a set of vectors is linearly independent, we cannot represent one in terms of the others

- If a set of vectors is linearly dependent, as least one can be written in terms of the others

Dimension

- Dimension of a space
  - In a vector space, the maximum number of linearly independent vectors is fixed
- Basis
  - In an *n*-dimensional space, any set of n linearly independent vectors form a *basis* for the space
- Given a basis $v_1, v_2, \ldots, v_n$, any vector v can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

where the $\{\alpha_i\}$ are unique

Representation

- Need a frame of reference to relate points and objects to our physical world.
  - For example, where is a point? Can’t answer without a reference system
  - World coordinates
  - Camera coordinates

Coordinate Systems

- Consider a basis $v_1, v_2, \ldots., v_n$
- A vector is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots. + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \ldots. \alpha_n\}$ is the *representation* of v with respect to the given basis
- We can write the representation as a row or column array of scalars

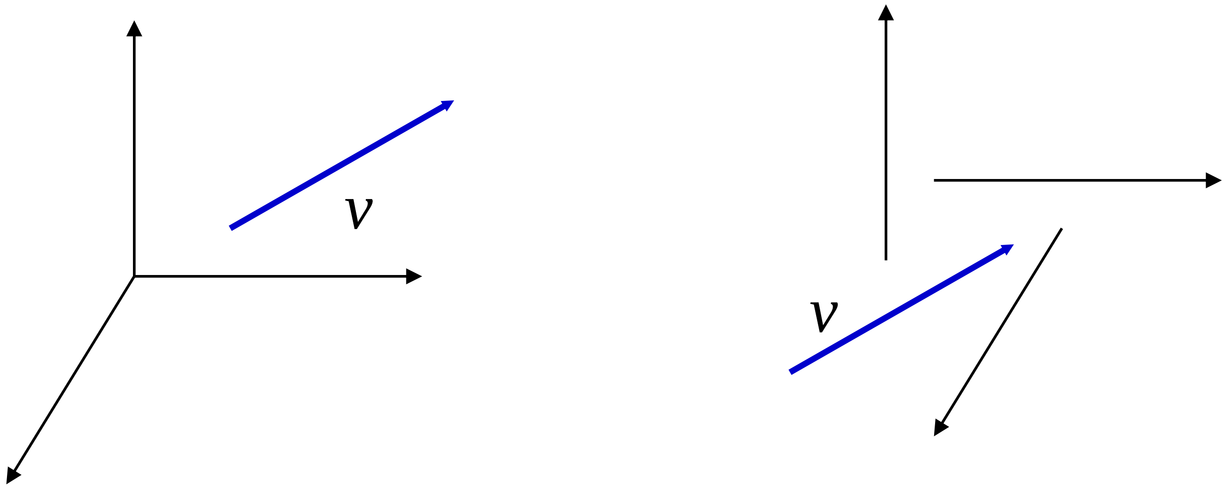
$$\mathbf{a} = [\alpha_1 \ \ \alpha_2 \ \ \ldots. \ \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ \alpha_n \end{bmatrix}$$

Example

- $v=2v_1+3v_2-4v_3$
- $\mathbf{a}=[2\ 3\ -4]^T$
- Note that this representation is with respect to a particular basis

Coordinate Systems

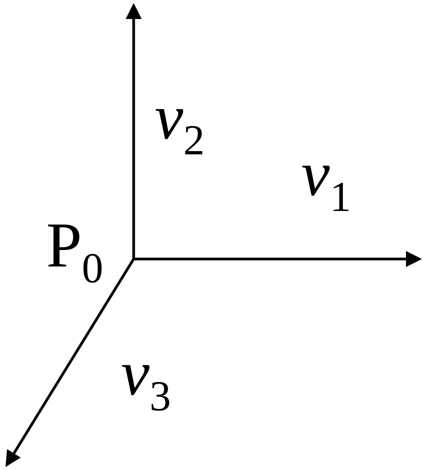
- Which is correct?



- Both are because vectors have no fixed location

Frames

- A coordinate system is insufficient to represent points
- If we work in an affine space we can add a single point, the *origin*, to the basis vectors to form a *frame*



Representation in a Frame

- Frame determined by (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as

  $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

- Every point can be written as

  $P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

Confusing Points and Vectors

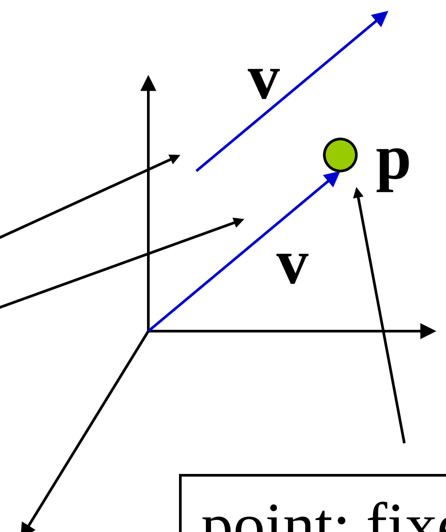
Consider the point and the vector

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

They appear to have the similar representations

$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3]$ $\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3]$



Vector can be placed anywhere

point: fixed

A Single Representation

- If we define $0 \bullet P = \mathbf{0}$ and $1 \bullet P = P$ then we can write

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]\ [v_1\ v_2\ v_3\ P_0]^{\mathrm{T}}$$

$$P = P_0 + \beta_1 \mathrm{v}_1 + \beta_2 \mathrm{v}_2 + \beta_3 \mathrm{v}_3 = [\beta_1\ \beta_2\ \beta_3\ 1\ ]\ [v_1\ v_2\ v_3\ P_0]^{\mathrm{T}}$$

- Thus we obtain the four-dimensional ***homogeneous coordinate*** representation

$$\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]^{\mathrm{T}}$$

$$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3\ 1\ ]^{\mathrm{T}}$$

Homogeneous Coordinates

- A three dimensional point $[x\ y\ z]$ is given as
  $\boldsymbol{p} = [x'\ y'\ z'\ w]^{T} = [wx\ wy\ wz\ w]^{T}$
- We return to a three dimensional point (for $w \neq 0$) by
  $x = x'/w$; $y = y'/w$; $z = z'/w$
- If $w=0$, a vector.
- Homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions.

Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
  - Hardware pipeline works with 4 dimensional representations
  - For orthographic viewing, we can maintain $w=0$ for vectors and $w=1$ for points
  - For perspective we need a *perspective division*

Change of Coordinate Systems

- Consider two representations of a the same vector with respect to two different bases. The representations are

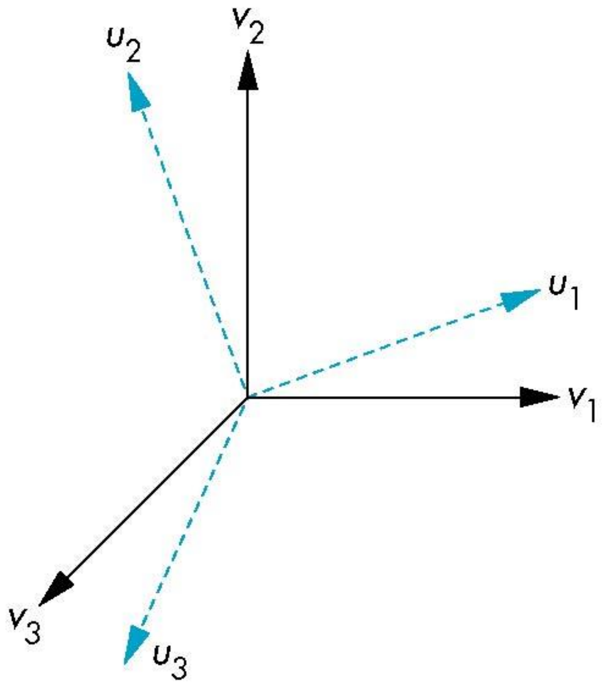
$$\mathbf{a}=[\alpha_1\ \alpha_2\ \ \alpha_3\ ]$$
$$\mathbf{b}=[\beta_1\ \ \beta_2\ \ \beta_3]$$

where

$$\mathrm{v}=\alpha_1 v_1+\ \alpha_2 v_2\ +\alpha_3 v_3 = [\alpha_1\ \alpha_2\ \alpha_3]\ [v_1\ v_2\ v_3]^{\mathrm{T}}$$
$$=\beta_1 u_1+\ \beta_2 u_2\ +\beta_3 u_3 = [\beta_1\ \beta_2\ \beta_3]\ [u_1\ u_2\ u_3]^{\mathrm{T}}$$

Representing second basis in terms of first

- Each of the basis vectors, u1,u2, u3, are vectors that can be represented in terms of the first basis



$$u_1 = \gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$$